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The manuscript should be typewritten on one side of the paper only, with wide margins and be double spaced throughout including titles, footnotes, literature citations and legends. Symbols, formulae and equations must be written clearly and with great care. Scientific names of genera and species are printed in italics and should be underlined in the typescript. Too many tables, graphs, etc. should be avoided. Each table should be typed on a separate sheet with its proper position marked in the text in pencil.

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# The Indian Academy of Sciences

At the invitation of this University the Indian Academy of Sciences is holding its 26th Annual Session at Madras on the 27th December, 1960, under the distinguished Presidentship of Professor Sir C. V. Raman, Kt., F.R.S., N.L., after a lapse of twenty-two years. The 1938 session was a memorable one from many points of view. With the great advances in science since then and the many outstanding contributions made by the Fellows of the Academy, we look forward with great interest and expectation to an even more memorable session.

Between 1938 and now it has been our privilege to welcome Professor Raman on many occasions, especially in 1957, during the Centenary Celebrations of this University when he presided over the scientific deliberations. We are fortunate that Professor Raman, a distinguished alumnus of this University, has given us yet another opportunity to have him in our midst. We extend to him and the Fellows and Delegates of the Academy attending this Session a most cordial welcome.

The Indian Academy of Sciences was incorporated at Bangalore in the year 1934. The purpose of the Academy has been to bring together a group of scientific men representative of the whole field of science and of all India to co-operate in promoting the progress of science and furthering the cause of scientific research. Soon after the establishment of the Academy, Maharaja Krishnaraja Udayar on the advice of Sir Mirza Ismail, who was then Dewan of Mysore, made a gift to the Academy of a site of eleven acres of land in a magnificent situation adjoining the palace, thereby enabling its headquarters to be permanently located at Bangalore.

The Academy as at present organised consists of a body of Fellows elected to it on the basis of their scientific achievements. The maximum number of the ordinary Fellows has been fixed at 250 and of the Honorary Fellows at 60. The latter are chosen from amongst the most eminent men of science in other countries. The affairs of the Academy are administered by a Council which is elected triennially from amongst its Fellows. The Council consists of the President, four Vice-Presidents, two Secretaries, a

Treasurer and fifteen members. The elections to the Council are made in such manner that while ordinary business of the Academy can be carried on efficiently at its headquarters, the opinion of Fellows resident elsewhere can make itself felt in its administration. To ensure this, the Annual General Meetings of the Academy at which elections of Fellows, Honorary Fellows and of the Council are made are usually held at one or another of various academic centres in different parts of India. The cities at which the Annual Meetings of the Academy have been held in the years past make a fairly long list. In alphabetical order they include, Ahmedabad (1), Allahabad (1) Annamalainagar (1), Bangalore (5), Baroda (1), Belgaum (1), Bombay (2), Cuttack (1), Delhi (1), Hyderabad (3), Madras (1), Nagpur (1), Poona (1), Tirupati (1), Trivandrum (1), Udaipur (1), and Waltair (2), the number of meetings held at each centre being indicated in brackets.

The Annual Meetings of the Academy also serve to bring the Fellows resident in different parts of India in personal touch with each other and enable them to exchange ideas with each other regarding subjects of common interest. At these meetings the President delivers his annual address which invariably deals with some scientific topic relating to his own researches. In the following days, other scientific addresses are delivered which are adapted to a general audience.

One of the main activities of the Academy since its establishment has been the publication of its monthly "Proceedings" which is a research periodical appearing in two series, 'A' and 'B', division between them being broadly that of the physicomathematical and biological sciences, respectively. The "Proceedings" have appeared without break or delay on the last date of every month since the establishment of the Academy. The total number of complete half-yearly volumes so far issued is 51. They form a veritable library of science covering a great variety of subjects, no less than 4,351 individual papers have so far been published making up a total of 41,600 printed pages besides numerous text figures and half-tone illustrations. These papers represent the original contributions to science of the Fellows of the Academy and of their students and collaborators. The 'Proceedings" have served as a forum for the publication of the scientific work of a whole generation of scientific workers. They have a fairly extensive circulation both in India and abroad and are regularly noticed in the bibliographical journals of the world.

The annual subscriptions from Fellows and the sale of the "Proceedings" form the financial mainstay of the work of the Academy. The work of the Academy has also received generous recognition and support from the Governments of India, Mysore, Kerala, Hyderabad, Bombay and Madras, and the Universities of Madras, Andhra, Annamalai and Patna and from the Indian Institute of Science. The Offices of the Academy are located in a modest little building on the Tumkur Road, in the site originally gifted to the Academy. This building as well as the residential quarters allotted to the Assistant Secretary in charge of the office were built by the Academy from its own resources supplemented by donations made by its Fellows.

EDITOR

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# An Account of the Life History and Feeding Habits of a Sole (Solea Elongata Day)*

BY

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(Received for publication on 27-10-1959)

ABSTRACT

The various stages in the life history of *Solea elongata* (Day) are described with the observations on its food and feeding habits during these stages. Feeding begins invariably with the transformation of the pro larvae into post larvae. The fish begins on a diet of cirripede nauplii, copepod nauplii, diatoms and algae during the early larval stages, but gradually changes over to larger Pelagic organisms such as copepods, larval bivalves polychaete larvae, decapod larvae, Lucifer, etc, as it grows and finally subsists on benthic organisms such as dead fish, mud dwelling polychaetes, foraminfera remains, etc.

## *Introduction*

The food and feeding habits of the economically important flat fishes of the temperate waters have been studied by Scott (1895-1922), Dannevig (1897), Petersen (1894), Clark (1920), Lebour (1922) and Shelbourne (1953). On the other hand very few Indian flat fishes have any economic value with the exception of *Cynoglossus sp.* of the Malabar coast which has been dealt with by Davanesan and Chidambaram (1948) and (1953). The food of *Cynoglossus sp.* of the estuaries of Bengal and tidal zones of Hooghly has been briefly referred to by Mookerjee, Ganguly and Mazumdar (1946), Hamilton (1822), Mookerjee (1946) and Jones and Menon (1951).

Whereas the life histories of the flat fishes of temperate waters have been studied by Kyle (1898, 1903, 1913 and 1921), Holt (1893), Cunningham (1896), Ehrenbaum (1909), Norman (1927, 1934), Townsend (1936), Rapson (1940), Clark (1914), Budd

* Part of this approved for the Ph.D. degree, University of Madras.

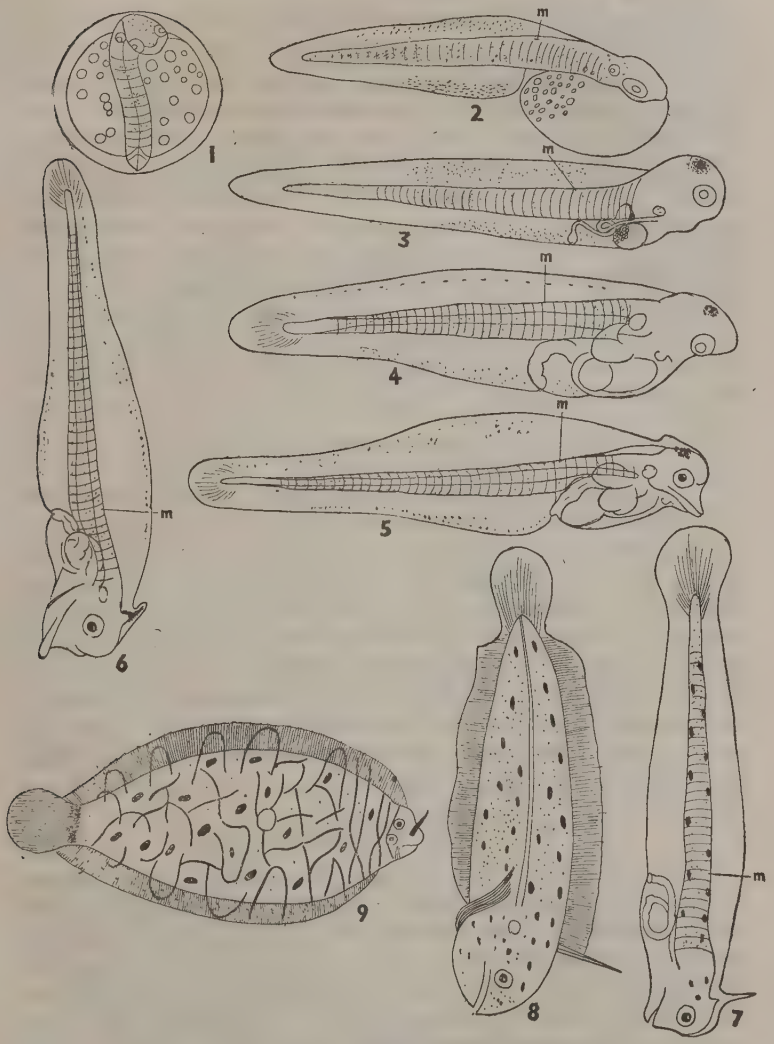
(1940), etc., we have practically no account of even a single Indian form. Some authors have dealt with the eggs while some others with larvae of one or the other of the species (vide Gopinath 1946, John 1948, 1951, Jones and Menon 1951, Nair 1952, Vijayaraghavan 1953, Seshappa and Bhimachar 1955, etc.). Recently the complete life history of *Cynoglossus lingua* from the eggs to the juvenile stage was studied with special reference to their feeding habits during different stages of growth by hatching the eggs and rearing the larvae and by feeding them with different items of food by the author (Kuthalingam 1957) where in the method used in rearing the fish eggs and larvae is described in details.

### *Eggs*

A total number of 108 eggs were collected from the plankton hauls made in February, 1956. The eggs were easily distinguished from the eggs of other fishes by the presence of a lage number of oil globules. The eggs were perfectly globular, comparatively small, and measured 0·7 mm. in diameter. The oil globules were about twenty-five in number and of different sizes, and were crowded more towards the head region of the embryo. The yolk was clear and colourless. The eggs were fairly advanced in development when they were collected (Fig. 1). The eye and the auditory vesicles were well formed and the heart was beating. Black pigment spots were noticed on the embryo. Myotomes were faintly seen. The embryo made occasional twisting movements and about 8-45 A.M. the next day the larvae hatched out of the eggs.

### *Larvae just hatched* (*Fig.* 2)

The larvae, measured 1·6 mm., were transparent, and floated in water showing very little movement. The head was sunk into the yolk mass and small oil globules were crowded near the posterior side of the yolk mass away from the head region. Eyes were unpigmented, and the auditory vesicle was seen in front of the most anterior myotome. There were eleven pre-anal myotomes. There was no indication of the mouth. The fins were present as fin-folds and they were continuous and entire. There was a faint indication of the pectoral fin between the 2nd and 3rd myotomes. Black pigment cells were found along the margin of the ventral fin-folds behind the anus. Similar pigmentation was seen along the middle of the dorsal fin-fold. Minute black pigment spots were seen distributed in the body of the embryo.




TEXT FIGS. 1-9: Larval stages of *Solea elongata* Day.
FIG. 1. Egg in an advanced stage of development; FIG. 2. Larva just hatched; FIG. 3. Larva 10 hours after hatching; FIG. 4. Larva 24 hours after hatching; FIG. 5. Larva 32 hours after hatching; FIG. 6. Larva 48 hours after hatching; FIG. 7. Larva 72 hours after hatching; FIG. 8. Larva 16 days after hatching; FIG. 9. Larva 18 days after hatching.

*Larvae* 10 *hours after hatching* (*Fig.* 3)

The length of the larvae had increased to 2·1 mm. The yolk was much reduced, the oil globules were fewer and smaller and arranged close together to form a small group on the posterior side of the yolk. There was a reduction in the number of pre-anal myotomes from 11 to 9 and the short coiled alimentary canal opened to the exterior below the nineth myotome and behind the anus, so that, twenty-three post-anal myotomes were counted. A slight projection at the anterior ventral region of the head indicated the future oral area. The eyes were still unpigmented. Black pigment cells were found along the margin of the ventral fin-fold in front as well as behind the anus. The pigmentation in the dorsal fin-fold remained the same as in the previous stage. A black patch of pigment cells was noticed above the eye on the dorsal side of the head.

*Larvae* 24 *hours after hatching* (*Fig.* 4)

The length had increased to 2·2 mm. The yolk was much reduced in size and the oil globules could no more be distinguished The pigment cells on the fin-folds had become fewer while ramified yellow pigment cells appeared on the yolk behind the eyes and on the dorsal side of the head, and similar yellow cells were arranged to form three equi-distant incomplete rings round the body as indicated in the figure. Developing fin-rays became faintly visible in the caudal fin-fold and the pectoral fins appeared as transparent fin-folds below the auditory vesicle. But there was no indication of the mouth and there were nine pre-anal myotomes and twenty-nine post-anal myotomes.

*Larvae* 32 *hours after hatching* (*Fig.* 5)

The larvae which had increased in size by 2 mm. had now become post-larvae. The mouth was formed, the alimentary canal had become coiled, and the anus had shifted forward to the 8th myotome. The pectoral fins had appeared as thin transparent semi-circular flaps. The dorsal fin showed a slight wedge-shaped depression anteriorly indicating the development of future tentacle. Fin-rays appeared in the caudal region. The eyes were pigmented and there was no change in the general colouration of the body.

The larvae did not feed though several types of food items were offered to them.

*Larvae 48 hours after hatching (Fig. 6)*

The larvae measured 2·6 mm. But for the development of the tentacle at the beginning of the dorsal fin-fold, the prominent lower jaw, and the disappearance of the pigmentation over the antero-dorsal side of the head, the post-larvae were similar to the thirty-two hours old post-larvae.

The larvae did not feed, and many of them died at this stage.

*Larvae 72 hours after hatching (Fig. 7)*

On the morning of the fourth day the larvae were moving about here and there inside the glass containers. They measured 3·2 mm. in length. The eyes were pigmented. The mouth was formed and opened into the intestine which was coiled as in other flat fishes. There was no change in the number of myotomes, but the tentacle was well developed. The black pigment cells on the dorsal and ventral fin-folds were grouped to form large black blotches. The yellow pigment cells disappeared completely, but, the black pigment cells on the body of the larvae remained as before.

The larvae began to feed only on the morning of the fourth day of hatching and the food consisted of cirripede nauplii of *Balanus amphitrite,* copepod nauplii of *Macrosetella sp.* diatoms *Coscinodiscus sp. Triceratium sp.* and *Pleurosigma sp.,* algae *Trichodesmium sp,* larval bivalves, and polychaetes. Adult copepods which were also introduced into the vessel containing the post-larvae, were rejected.

During the next seven days there was no change in the diet of the post-larvae which were supplied with plankton collected each day.

The larvae grew rapidly and on the 12th day they measured 9·2 mm. They were now large enough to feed on adult copepods such as *Oithona spinulosa, Acartia erythraea, Euterpina acutifrons, Paracalanus parvus, Eucalanus sp,* and *Corycaeus sp.* Also included in their diet were larval bivalves, polychaete larvae, decapod larvae, megalopa stages of brachyura and *Lucifer,* rejecting diatoms and algae which formed their food previously. To ascertain if algae and diatoms were deliberately avoided, a few larvae were fed with only diatoms (*Asterionella sp., Coscinodiscus sp. Triceratium sp, Pleurosigma sp.*) and *algae Trichodesmium sp.* together with smaller number of adult copepods and other plank-

tonic oganisms with a few larvae in a watch glass. On examination under binocular it was found that the stomach did not contain any vegetable matter and that the larvae fed only on copepods and other animals. Though entirely carnivorous, the larvae did not feed on particles or dry fish soaked in water.

There was no change in diet during the next three days.

*Larvae 16th day after hatching* (*Fig.* 8)

The larvae were actively swimming about on the morning of the 16th day. They measured 11·2 mm. and appeared broad with the dorsal and the anal fins confluent with each other. The first two rays of the dorsal fin-fold were elongated as a finger-like process. The anal fin was rounded. The mouth was wide with pointed teeth in both the jaws. The eyes were small and visible from the right since the the left eye approximatly close to the dorsal side of the head. The whole body except the region of the fins was opaque and therefore the internal organs could not be made out. The whole body was covered with black spots and black blotches. The larvae which were active in the morning became less active towards evening. They rested at the bottom of the vessel and though the vessel contained several items of food, the larvae did not feed until the evening of the 17th day.

*Larvae 18th day after hatching* (*Fig.* 9)

The larvae metamorphosed into juveniles, the adult characters of the species were recognised, and the fin formula was as follows: D.73.P.9.V.4 A.61.C.18.L.1.120. From time to time the juvenile rested flat on the bottom or struck against the sides of the container. Both the eyes were situated close to each other on the right side of the head. The body was compressed dorsoventrally with a broad annterior and a tapering posterior end. Two nostrils were seen, one situated in front of the lower jaw, the other smaller and tubular one situated anterior to the former. The juveniles were of olive tone in colour, with black spots and black blotches all over the body. All the taxonomic features were in evidence and the fish could be identified as *Solea elongata.*

Supplied with fresh concentrated plankton samples, it was found that the juveniles fed on a wide range of animals such as *Acetes, Lucifer, Mysis, Squilla-* larvae, larval-, *Penaeus,* cumacea, decapods, crustacean eggs, copepods, cirripede larvae, megalopa of brachyura, *Sagitta* and molluscan larvae. *Lucifer, Mysis* and

*Acetes* formed the main items of the diet of the juveniles. The juveniles, however, also attacked animals as large as themselves, biting off small pieces. Smaller-sized prey, like the copepods, were not preferred as they were done a week ago.

When the Juveniles were supplied with fish eggs and larvae from the plankton, they were totally rejected. Similarly, when they were given diatoms and algae together with other planktonic animals such as *Acetes, Lucifer* and *Cumacea,* it was found that none of them had vegetable matter in their stomachs, suggesting a continuance of the carnivorous habit.

### *Food of adult fishes*

During May, August, September and December, nearly sixty-eight adults ranging from 120 mm.—140·5 mm. in length were obtained from the landings at Madras. Analysis of the stomach contents showed that molluscan shells, foraminifera, decayed dead fish, mud-dwelling polychaetes as well as their tubes, fish bones, and scales were present. The quantity of bottom mud found in the stomach was considerably large. In two fishes the entire stomach was filled with only mud. It is probable that *Solea elongata* subsists on bottom living animal and organic matter present in the bottom mud. Since sexually mature specimens have never been collected in Madras, it is presumed that the species breed somewhere off the coast of Madras.

### *Summary*

1. Previous work on the life histories of fishes, the feeding habits of larvae, juveniles done both in India and abroad is briefly reviewed.

2. Detailed account of the complete life history of the fish with description of eggs, hatched larvae, post larvae and juveniles of *Solea elongata* is given.

3. The different stages in the life history of *Solea elongata* are described, with observations on its food and feeding habits during these stages. Feeding begins invariably with the transformation of the pro-larvae into post-larvae. The fish begins on a diet of nauplii of cirripede and copepod during the larval stages, but gradually changes over to larger pelagic organisms such as *Acetes, Mysis, Squilla,* etc., as it grows and finally subsists on Benthic organisms.

*Acknowledgements*

I wish to thank Prof. C. P. Gnanamuthu, M.A., D.Sc., Director, University Zoology Research Laboratory, Madras for suggesting the problem and for his guidance throughout the course of investigation. I am indebted to Dr. G. A. Steven, the laboratory Citadel Hill Plymouth and Dr. M. V. Lebour for offering their valuable suggestions.

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# Lilavati—Constructional Details of 'Digito-Analogue' Model III and its Applications*

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ABSTRACT

This paper deals with the constructional details of the 'digito-analogue' Model III of Līlāvati—a computer for solving simultaneous and secular equations. Several iterative processes for solving both these equations are mechanised and the programming of these methods are elucidated with examples. In addition, the applications of this computer for the inversion and the factorisation of matrices are indicated. This model has three figure accuracy and it is interesting to state that it contains no relays of any sort.

## 1. *Introduction*

The essential principles involved in the design of the "digito-analogue" Model III of Līlāvati were described in the previous part [4] (G. N. Ramachandran and E. V. Krishnamurthy, 1959 — Hereafter referred to as Part III). This paper is devoted to the constructional details of the six-equation computer, built in the laboratory, based on the above principles. The programming of problems, operation and other applications of the computer are also described.

Before actually going into the description of the computer and its operation, the details of the various circuit arrangements will be given. As in the previous parts, we shall denote the set of simultaneous equations by

$$\sum_{j=1}^{n} a_{ij}x_j = g_i \ (i = 1 \text{ to } n) \qquad .. \quad (1)$$

and the set of secular equations by

$$\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i \ (i = 1 \text{ to } n) \qquad .. \quad (2)$$

*Part IV of Lilavati series. The references to Parts I, II and III are given in (2), (3), (4).

### 2. *The Step selector*

In order to mechanise the various iterative processes described in Part III, it is enough if we set up the analogy of a single equation at a time. The way in which the connections in each Kelvin-Varley bridge (say the $j$th) can be made by means of three decade switches so that $E_{out}$ corresponds to $a_{ij}$ times $E_{in}$ was indicated in Part III (section 7, Figs. 4 & 5). Fig. 5 of that part corresponds to $j = 1$, and if $A_{i1}$ is connected to $P_1$, $A'_{i1}$ to $P'_1$, $B_{i1}$ to $Q_1$ and

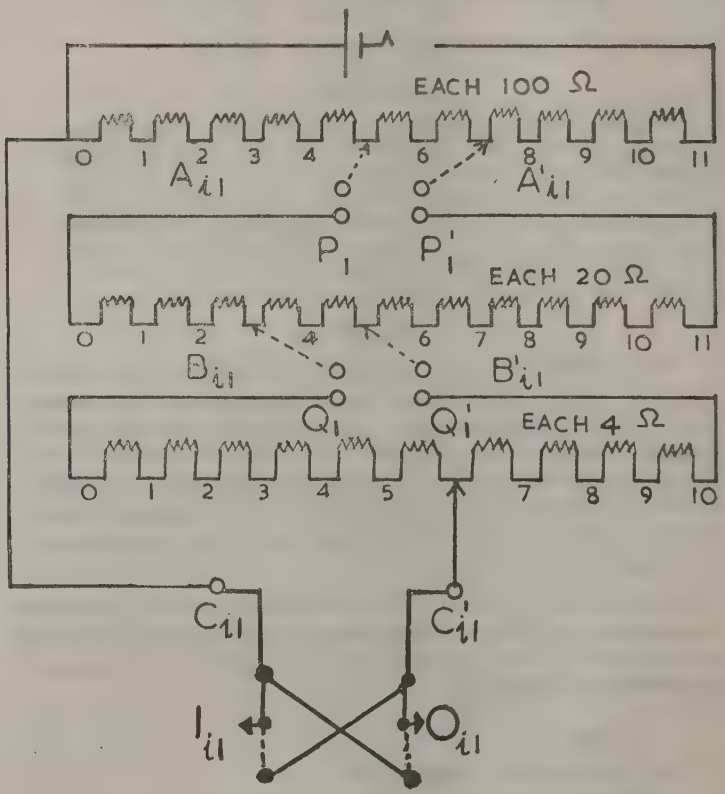


FIG. 1. Analogy of $\pm a_{i1}\ x_1$ from a single Kelvin-Varley bridge

$B'_{i1}$ to $Q'_1$, and current $x_1$ flows through this box, then the output across $C_{i1}$ and $C'_{i1}$ would be $a_{i1}x_1$ but for the sign. In order to obtain $a_{ij}x_j$ for different values of $i$, it is only necessary to make,

in turn, the above interconnections for each value of $i$, i.e., for each of the rows, one at a time. The way in which this is done is shown in Fig. 1.

We shall indicate the poles of the three switches occurring in the $i$th row and $j$th column by $A_{ij}$, $A'_{ij}$; $B_{ij}$, $B'_{ij}$ and $C_{ij}$ respectively. Suppose now that the point $A_{ij}$ is connected to $P_j$, $A_{ij}$ to $P'_j$, $B_{ij}$ to $Q_j$ and $B'_{ij}$ to $Q'_j$. (The connections are indicated for the first box (i.e. $j = 1$) and the $i$th row in Fig. 1). Now, suppose that all the three decade switches A, B, C at each position $ij$ are set to correspond to the three digits of the coefficient $a_{ij}$ in Eqn. (1) and the above connections $A_{ij}$, $A'_{ij}$, $B_{ij}$ and $B'_{ij}$ are made respectively to $P_j$, $P'_j$, $Q_j$ and $Q'_j$ for the various values of $j$. Then it is obvious that, if currents $x_j$ are made to flow through the various boxes, we have between the points $C_{ij}$ and $C'_{ij}$ voltages $|a_{ij}x_j|$, which are just the terms of the $i$th equation. Connected through a plus-minus switch, as in Fig. 1, the voltage across $I_{ij}$ and $O_{ij}$ would be equal to $a_{ij}x_j$ with proper sign. These could then be connected up in series and we then have a complete analogy of Eqn. (1) which in matrix notation is

$$AX = G.$$

Obviously, with suitable changes, it can also represent the equation

$$AX^{(p)} - G + R^{(p)} = 0. \qquad \ldots \quad (3)$$

Since the sign of each coefficient can be a plus or a minus, there must be a plus-minus switch associated with each set of three switches for setting $a_{ij}$.

Since the $x'_j$s which occur in every one of the equations are the same, it is only necessary to have a provision, whereby the necessary voltages $a_{ij}x_j$ ($j = 1$ to $n$) are tapped from each of the boxes corresponding to one row at a time, and these voltages connected together in series, in order to have the analogy of each one of the equations (1) in turn. This is done by means of a 'step selector' switch.

(a) *Selection of* $a_{ij}x_j$

The circuitry of the step selector switch which taps out the voltages $a_{ij}x_j$ from the various boxes is shown in Fig. 2. The poles of the various set-switches A, A', B and B', are connected to the ways of switches $P_j$, $P'_j$, $Q_j$ and $Q'_j$. The 4 $(n+1)$

switches $P_j$, $P'_j$, $Q_j$ and $Q'_j$ ($j = 1$ to $(n+1)$) are all ganged, and their poles go to the corresponding box ($j$th box) as shown in Fig. 1, in which the connections are indicated for the first box ($j = 1$). Thus, if this system of ganged switches is set at the $i$th position, it is obvious that voltages $a_{ij}x_j$ would be available between the terminals of the plus-minus switch, which may be indicated by $I_{ij}$ and $O_{ij}$. (In Fig. 1, the inlet and outlet points $I_{i1}$ and $O_{i1}$ of box 1, are marked.)

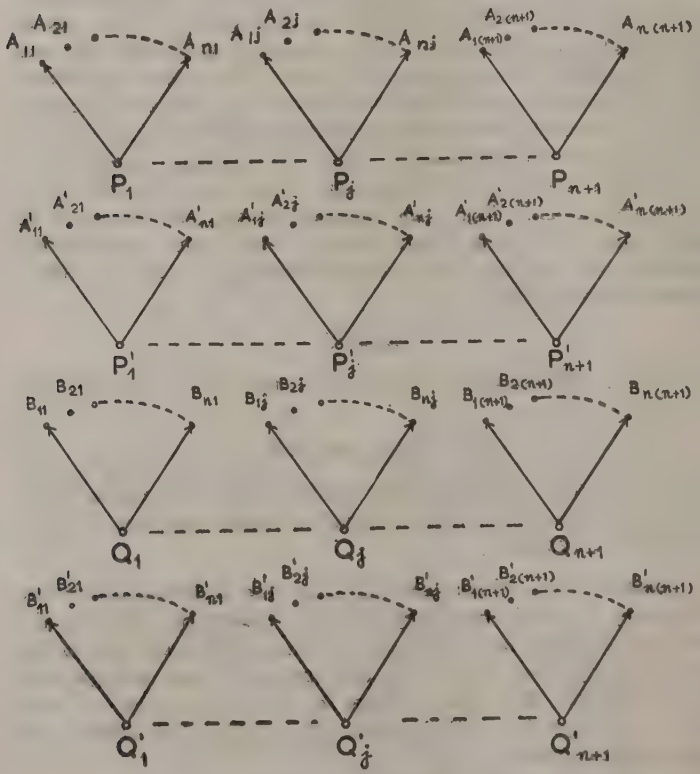


FIG. 2. Step selector connections for obtaining $a_{ij}$ $x_j$

It is to be noted that the total resistance of the potentiometer across the input terminals (end terminals of the 100 Ω series) is a constant independent of the setting of the various decade switches or of the step selector, neglecting the contact resistance

errors in the current-bearing circuit. The contact resistance had a value between 0·005 Ω to 0·05 Ω per contact, for the ordinary radio band-switches used in the instrument, and therefore did not affect the 0·1 per cent accuracy of the resistance. Further, 22 gauge wires were used for making the interconnections between the various points and their total resistance was small enough not to affect the three figure accuracy of the computer.

(b) *Addition of voltages in a row:*

Once the voltages $a_{ij}x_j$ are simulated from the various columns, the problem now reduces to the addition of these voltages corresponding to a row to get

$$\sum_j a_{ij}x_j - g_i = 0 \quad (i = 1 \text{ to } n)$$

For this purpose, the voltage output terminal of each one of the plus-minus switches must be connected to the input terminal of the next switch. Thus for the addition of voltages in the $i$th row, interconnections $O_{i1}$ to $I_{i2}$, $O_{i2}$ to $I_{i3}$,.....$O_{in}$ to $I_{i(n+1)}$ must be made and the added up sum will be available between terminals $I_{i1}$ and $O_{i(n+1)}$. If this voltage is to be made zero, a galvanometer has to be connected across these points to indicate null reading.

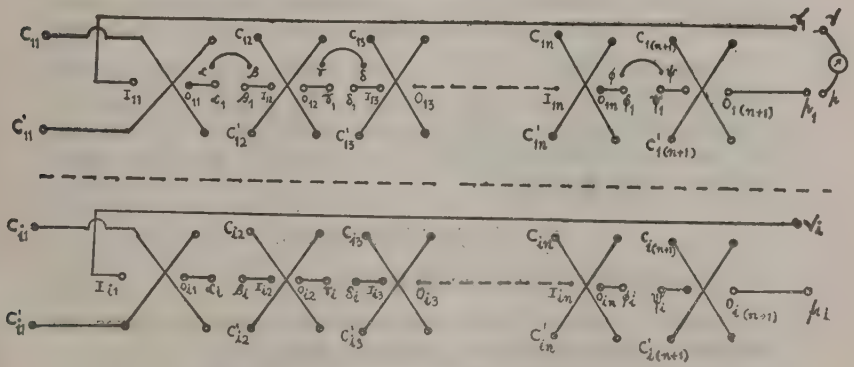

FIG. 3. Step selector connections for $\sum_j a_{ij}\, x_j$

The interconnections for each row and the connections for the galvanometer are all made by the same step selector which selects out the voltages $a_{ij}x_j$ as mentioned above. This is shown in Fig. 3. $(\alpha, \beta)$, $(\gamma, \delta)$,....., $(\varrho, \tau)$ and $(\varphi, \psi)$ are the poles of switches ganged to the above-mentioned 4 $(n+1)$ poles of the step selector

and each pair within brackets is shorted. $(\alpha_i, \beta_i)$, $(\gamma_i, \delta_i)$ ....., $(\varrho_i, \tau_i)$ and $(\varphi_i, \psi_i)$ are the $i$ $(=1$ to $n)$ ways of these poles and it is evident from Figs. (2) and (3) that when the switch is turned to each one of the ways from 1 to $n$, the voltages $a_i x_i$ and $g_i$ are all picked out, the corresponding terms in a row are added up and that the sum comes out at the output terminals $(\mu, \nu)$. If the galvanometer is connected across the poles $(\mu, \nu)$, the added sum can be checked for null value at each setting and thus each row of Eqn. (1) will be realised at a time.

Thus the step selector requires $[4(n+1)+2n]$ or 6 $(n+1)$ poles for the simulation of Eqn. (1). For the simulation of Eqn. (3), required in the RH process, $(n+2)$ columns are required and hence the step selector must have $[(n-2)+2(n+1)+2]$ or 6 $(n+2)$ poles. In the 6 Equation model constructed in this laboratory, 7 columns were provided for the simulation of a $6 \times 7$) matrix. In this way, 6 equations can be solved by the GS process and 5 by the RH process.

A ganged switch with 42 poles and 6 ways served as the step selector. Two more poles $(\omega, \xi)$ are added to this selector for controlling the pilot lamps indicating the appropriate current-adjusting potentiometers or the residual set switches $(r)$, as the case may be, depending upon the nature of the problem to be solved. This lighting up is controlled by the 'sequence selector'.

Several ordinary 10 pole—6 way radio band-switches were ganged and used as the step selector. The contact resistance of all these switches had a value between 0·005 Ω to 0·05 Ω per contact and were sufficiently low as to not impair the three figure accuracy of the machine.

### 3. *The feed-back selector*

(a) *Direct feed-back with multiplication:*

Another opeation to be performed in various problems, as described in Part III, is to set up the analogy of

$$X^{(p+1)} = \frac{1}{b} R^{(p)} \qquad .. \quad (4)$$

or, in component form,

$$x_i^{(p+1)} = \frac{1}{b} r_i^{(p)} \qquad .. \quad (5)$$

This occurs in the RH process for the solution of simultaneous equations and in the eigenvalue problems.

Eqn. (5) demands that the voltages read in the last column (the residual column, $n+1$) have to be multiplied by a scalar number $1/b$ and transferred to each one of the $n$ columns, viz. a fraction of the residual in the $i$th row, read in the $i$th element of the residual column, has to be fed back (as the $i$th unknown) in the $i$th column,, for each value of $i$ $(=1$ to $n)$.

In order to set up the analogy of Eqn. (5), a selector switch should perform three operations at a time.

(1) Picking out the voltages $r_i$ from the residual column,

(2) Multiplication by $1/b$,

and (3) Comparing the voltage $r_i \times 1/b$ (viz. the voltage across $a_{i,n+1}$ by passage of a current $1/b$) with that across unit resistance in the $i$th current circuit by the passage of current $x_i$. (viz. the voltage $x_i \times 1$).

As explained in the last section, the picking out of voltages from one of the columns requires 4 poles and $n$ ways. The comparison of voltages also requires 4 poles and $n$ ways. It is clear that the step selector has already the provision to do the first of these operations. But, in addition, it connects up the whole row in series. Therefore, if voltage comparisons or transfers have to be done, the interconnections existing between the residual column and the other columns have to be broken and the interconnection for comparison has to be made. This can be achieved using 4 more poles in the step selector and making a sequential change in the operation by means of the sequence selector, which disconnects the residual column from other columns and makes the connections for comparison. However in the present computer, it was not done in this way. It was found to be more convenient to have a separate switch called the 'Feed-back selector', for this purpose.

This selector performs operations (1) and (3), namely the picking out of $r_i$ and the comparison with the $i$th column. The operation (2) namely, multiplication by a scalar number $1/b$, is carried out by changing the current through the residual column (7th column), which is initially unity, to a new value $1/b$. Thus the voltages in this column now change to $r_i/b$. This change in current is easily achieved, if two current potentiometers are used for feeding in the 7th Kelvin-Varley resistance box, of

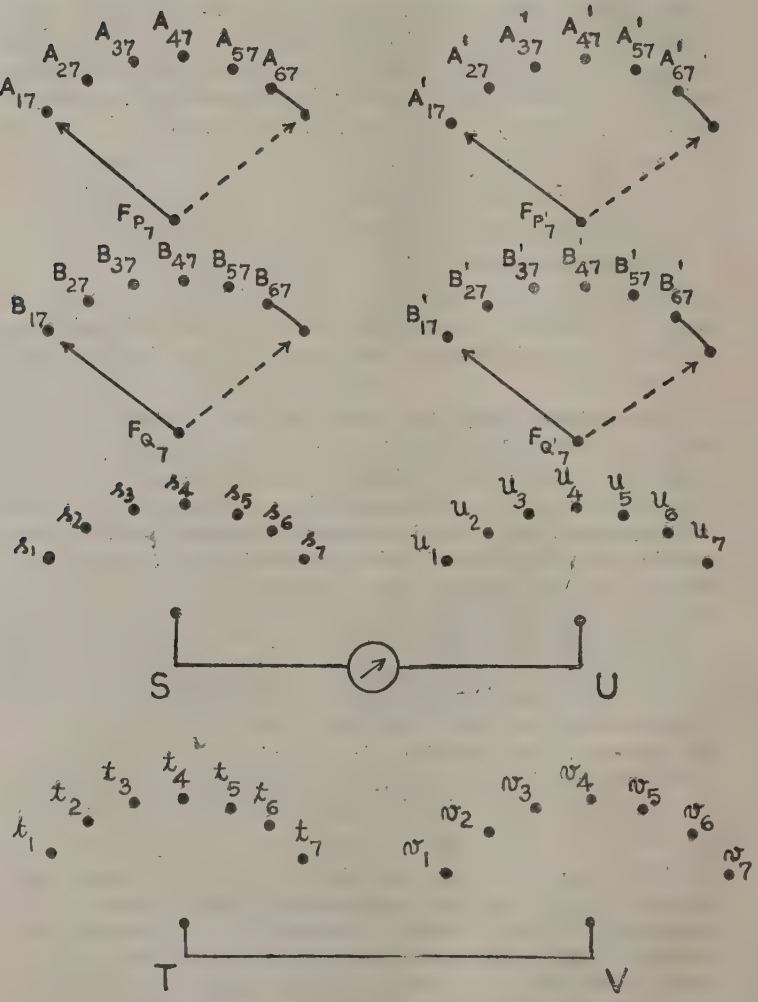


FIG. 4(a) Feed back selector connection for various settings 1 to $n$

whch one or the other can be connected, to send either unit current or a current $1/b$. In practice, it is more convenient to have the initial current as $b$ and so after multiplication by $1/b$, the new value becomes unity. In such a case either the $b$-current or the unit current potentiometer must be connected to the 7th box, while reading the residuals or while feeding back. This

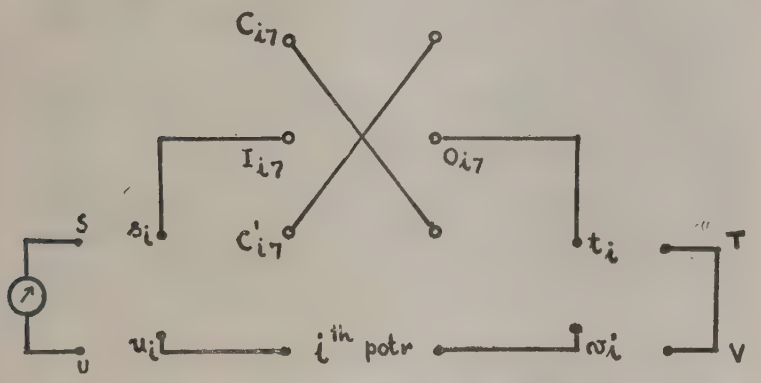


Fig. 4(b) Feed back selector connections for comparison of voltages

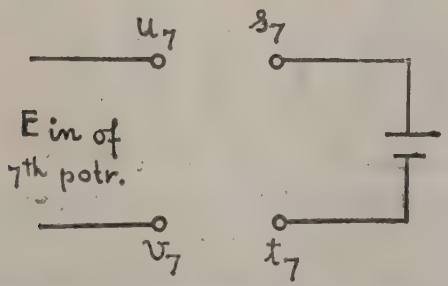


Fig. 4(c) Feed back selector connections for standardisation of the 7th column

operation is carried out by the 'sequence selector' (Fig. 5. c.). The end terminals of the 100 Ω series of the 7th box, namely its input terminals, form the two poles of the sequence selector. These poles in their first three ways select the $b$-current potentiometer and in the fourth, the unit current potentiometer, thus carrying out the multiplication by $1/b$, when the sequence selector is changed from its third position to the fourth.

A 10 pole 7 way switch was used as the feed-back selector. The circuit diagram of this is shown in Fig. 4(a).

The poles ($F_{P_7}$, $F_{P'_7}$), ($F_{Q_7}$, $F_{Q'_7}$) are connected respectively to the end terminals of the 20 Ω series and the 4 Ω series of the 7th resistance box, through the sequence selector ways ($d_2$, $e_2$), ($f_2$, $g_2$) and ($d_4$, $e_4$), ($f_4$, $g_4$) [Figs. 5(a) and (b)], viz. in the second and fourth positions of the sequence selector, the 7th box is connected to the feed-back selector.

The poles S and U are connected to the galvanometer through the ways ($h_2$, $i_2$) and ($h_4$, $i_4$) of the sequence selector. The poles T and V are shorted through the ways ($j_2$, $k_2$) and ($j_4$, $k_4$) of the sequence selector. The poles W and X are added to this switch for lighting up either the appropriate current adjusting potentiometers during feed back, or the residual set-switches during the matrix multiplication process. This is predetermined by the sequence selector. [Figs. 5(a) and (b)].

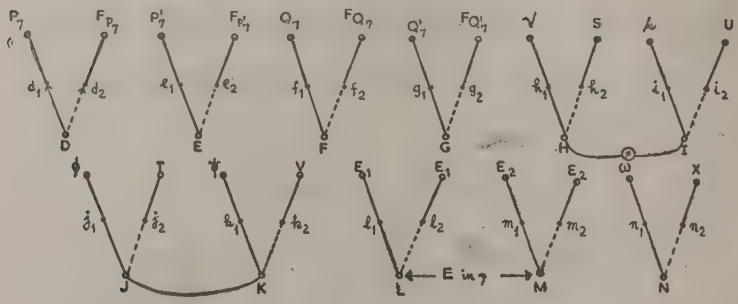


FIG. 5(a) Sequence selector connections at settings 1 and 2

It is therefore evident that when the sequence selector is in the second position, the feed back selector compares, in each one of its six ways, one of the voltages $a_{i7}$ ($i = 1$ to 6) by the passage of current $b$ through the 7th box with that across units resistance in the $i$th current circuit, by the passage of current $x_i$. If the sequence selector is in the fourth position, the feed-back selector compares in each one of its six ways, one of the voltages across $a_{i7}$ ($i = 1$ to 6) by the passage of unit current with that across unit resistance in the current circuit by the passage of current $x_i$. This is so, since in the fourth position of the sequence selector, the unit current potentiometer is connected across the 7th box input terminals, instead of the $b$-current potentiometer. In other

words, in the 4th position of the sequence selector comparison is done after multiplication by $1/b$.

The circuit diagram for the comparison of each one of the current circuits for $i = 1$ to 6 with the voltages $a_{i7} \times b$ or $a_{i7} \times 1$ is shown in Fig. 4(b).

(b) *Current measurement:*

In the settings 1 and 2 of the sequence selector, since the same $b$-current potentiometer is connected to the 7th box, the same current flows through the 7th column while balancing the equations by the Gauss-Seidel process and while comparing each one of the current circuit $i$ with each one of the voltages $a_{i7} \cdot b$. Therefore in the setting 2 of the sequence selector, the feed back selector can directly be used for measuring the unknowns (currents), as no change in scale factor is introduced. Thus the solution can be read off in this way, while solving by the GS process.

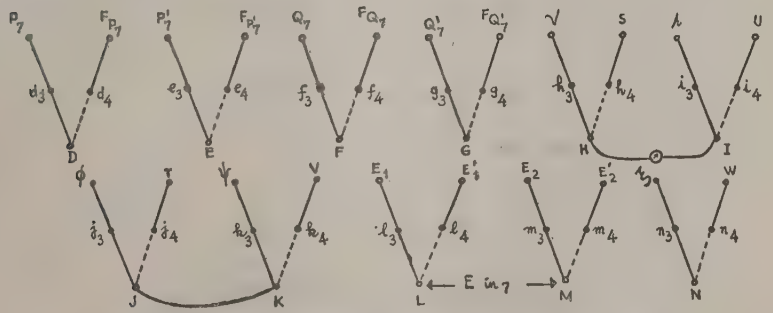


FIG. 5(b) Sequence selector connections at settings 3 and 4

(c) *Standardisation:*

If dry batteries are used, the currents through the various columns will not be steady and so, in practice, one of the columns has to be standardised occasionally, while the solution is in progress. Since the residual column is taken to be the standard and all the other columns are adjusted with respect to this, the former is standardised with reference to a steady source. A good accumulator was used in practice. This standardisation is done in the 7th way of the feed-back selector (Fig. 4c). Whenever required, therefore, the standardisation could be checked.

4. *The sequence selector*:

As already explained, the sequences to be performed in the computer are, in matrix notation,

(1) $A_{6x6}X - G = 0$ (GS Process)

(2) Reading currents X, with respect of Gth column, viz., reading off of the solution.

(3) $A_{5x5}X - G + R^{(p)} = 0$ or $A_{6x6}X - Y = 0$ (RH iterative process and seuclar equations or simple matrix multiplication).

(4) $X = \frac{1}{b} R^{(p)}$ or $X = -\frac{1}{\lambda} R^{(p)}$ (Transfer of data in RH method as well as in solving secular equations).

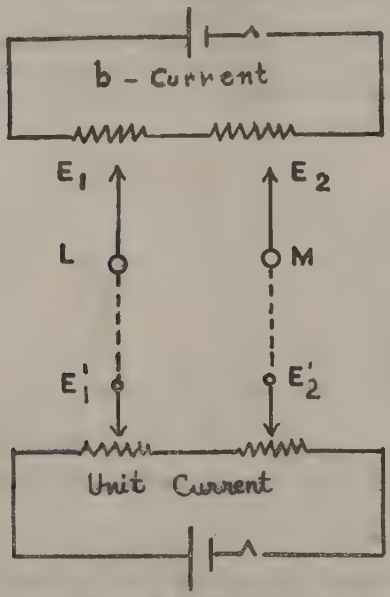


FIG. 5(c) b—Sequence selector connections current or unit current potentiometer

A 11 pole 6 way switch is sufficient to make the interconnections corresponding to one or the other of these processes. It has been designed so that the above sequences can be performed at settings 1, 2, 3 and 4 of the sequence selectors. The settings 5 and 6 are intended for the measurement of $\lambda$ and the setting of b, as will be explained below.

(a) *Sequences* 1 *and* 2:

In order to mechanise the Gauss-Seidel process, the residual column [$(n+1)$th or the 7th] is used for setting the $g_i$ values. For performing the iterative process, therefore, the 7th resistance box must be connected to the step selector and the interconnections for addition have to be made, bringing the galvanometer across the added voltage. The lamps corresponding to the current adjusting potentiometers must light up in this sequence.

In the second sequence, the residual column must be disconnected from the rest and the resistance box must be connected to the feed-back selector for measuring the current by comparison, as explained in Section 3(b). Since the same current flows through the residual column while balancing the equations and also while measuring the unknowns, no scale-factors are introduced and no standardization is required. However, if a dry battery is used, occasional checking up of the steadiness would be required (see Part II [2]). This is done in the 7th position of the feed-back selector, described earlier. In this sequence, the lamps corresponding to the residual set-switches must light up.

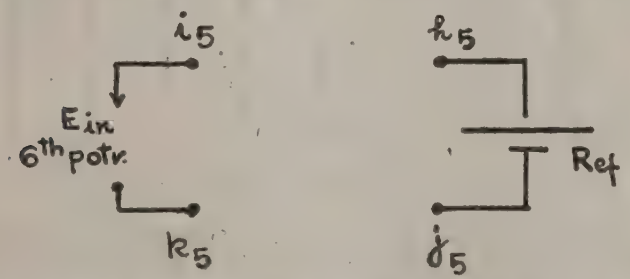


Fig. 5(d) Sequence selector connections for standardisation of the 6th column at setting 5

The circuit diagram for these two sequences is shown in Fig. 5(a). D, E, F and G are the poles connected to the 7th resistance box and it can respectively be connected to $P_7$, $P'_7$, $Q_7$ and $Q'_7$ in the first setting, and to $FP_7$, $FP'_7$, $FQ_7$ and $FQ'_7$ in the second setting. The galvanometer connected to the poles H and I can either be brought to the poles ($\mu$, $\nu$) of the step selector for addition or to the poles (S, U) of the feed-back selector for comparison. The poles (J, K) connect up the 6th column to the 7th column in the first setting and connect up the poles (T, V) for

comparison, in the second setting, as explained under feed-back selector. The poles (L, M) connect the input terminals of the 7th resistance box to the same $b$ current potentiometer in either of the settings. The pole N selects the pole $\omega$ of the step selector to light up the current-adjusting potentiometers, in the first setting of the sequence selector, and selects the pole X of the feed-back selector to light up the residual set-switches in the second setting.

(b) *Sequences 3 and 4:*

Sequence 3 is exactly similar to sequence I, except that the residual set-switches must be lighted, while operating the step selector, and hence the pole N selects the pole $\xi$ of the step selector.

The poles (L, M) connect the input terminals of the 7th resistance box to the $b$ current potentiometer, so that a current $b$ flows through the 7th column while reading the residuals. (Fig. 5 b).

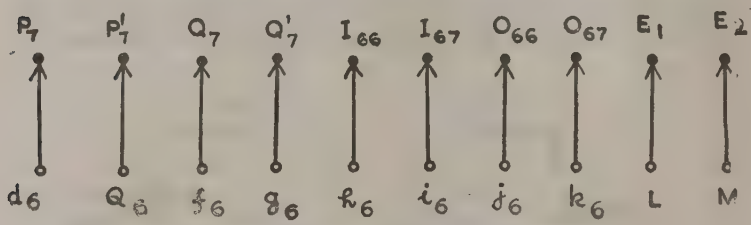


Fig. 5(e) Sequence selector connection at setting 6

In sequence 4, the poles (L, M) connect the unit current potentiometer across the input terminals of the 7th resistance box. (The terminals $E'_1$. $E'_2$ are the terminals of the unit current potentiometer) in Fig. 5(b) and 5(c)). Further the pole N selects the pole W of the feed-back selector now, to light up the current adjusting potentiometers during transfer. Thus comparisons are done after multiplication by $1/b$ in this sequence.

(c) *Sequence* 5:

Since the $g_i$'s are represented in the 6th column, while doing the RH process it is necessary to keep this column as a reference for the other columns. Therefore, the fifth setting of the sequence selector is intended for standardising the 6th column and for occasional check up for the steadiness. (See Fig. 5d).

(d) *Sequence* 6:

So far we have not explained either how the $b$ value is set in the seventh column or the value of $\lambda$ is read off. In this setting of the sequence selector these are done, by comparing the seventh column with the sixth, keeping the sixth as reference.

Fig. 5(e) shows this sequence. The 7th resistance box is connected now to the step selector through D, E, F and G and $b$ current potentiometer is connected across the input terminals of the 7th box. The interconnections between the 6th and 7th columns are broken as the points $j_6$ and $k_6$ are connected to $O_{66}$ and $O_{67}$ (Fig. 5 f) and not to the poles $\varphi$ and $\psi$ of the step selector. The galvanometer is connected between $I_{66}$ and $I_{67}$ so that the current through the 7th column can be set as $b$ or measured as $\lambda$, in the sixth position of the step selector.

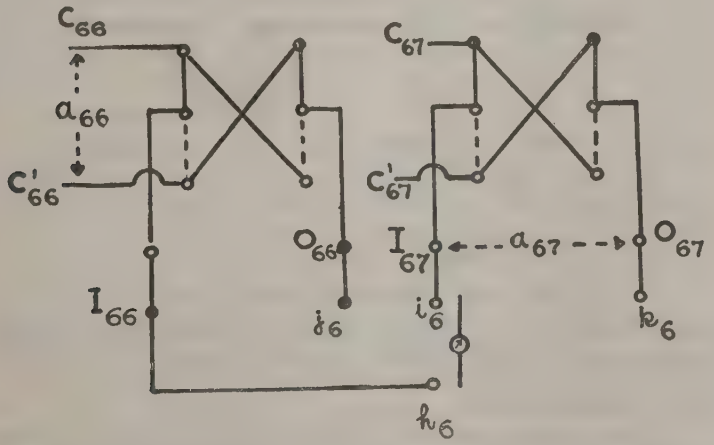


Fig. 5(f) Sequence selector connections for setting $b$ or measuring $\lambda$

Since $a_{66}x_6 = a_{67} \cdot b$, we have

$$b = \frac{a_{66}}{a_{67}}, \text{ since } x_6 = 1$$

Since both $a_{66}$ and $a_{67}$ have a range from 1 to 1000, any value of $b$ or $\lambda$ from $10^{-6}$ to $10^{6}$ can actually be measured. Of course this whole range is not necessary $\lambda$ or $b$ hardly exceeds 4 or 5 and rarely goes below 0·001 for a fifth order equation.

### 5. *Other details of the circuitry*

(a) *Current adjusting potentiometers and range switches*:

The current adjusting potentiometers used were ordinary 3-Watt wirewound volume controls with coarse and fine adjustments (100 Ω and 10 Ω). Since the maximum voltage applied to these is 10 V, this wattage is sufficient.

Since the minimum voltage across the output of these potentiometers could not be obtained to be less than 5 per cent of the input voltage, an extra range selector switch, with five settings and resistances connected in series across its terminals, was incorporated along with the on-off switch for each circuit. This is

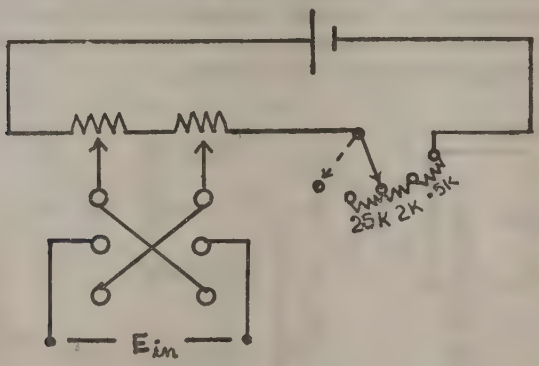


Fig. 6. Circuit of the range switch for currents

shown in Fig. 6. With this, a rangne of variation from 0·1 per cent to the full value of current could be obtained.

(b) *The sign switches*:

The sign switches used for currents and addition of terms were commercially available DPDT switches, as mentioned in earlier models, and need no particular mention here.

(c) *The galvanometer*:

A 30-0-30 student type Weston Galvanometer (resistance 100 Ω) with a full scale sensitivity of 660μA was used as the galvanometer. This sensitivity was quite sufficient for the range used. However, for lower current ranges it is desirable to use a better galvanometer and provision has been made for connecting an external galvanometer through a jack, when needed.

A 20 Ω resistance was used as the shunt, which can be removed when needed by a series switch. The galvanometer was always shunted by a 200 Ω resistance to suppress oscillations.

(d) *The resistance boxes:*

The value of the resistances in the three steps, namely, 100 Ω, 20 Ω, and 4 Ω of the Kelvin-Varley bridges was accurate to 0·1 per cent.

(e) *The batteries:*

Ordinary Eveready flash light cells (4·5 V) were used for all the columns except the 7th for which an accumulator was used. An accumulator also served as the reference standard (Plate I, Fig. 2).

6. *Description of the model*

Two photographs of Model III are given in Plate I (Figs. 1 and 2). Fig 1 is the front view showing the setting and operating switches, while Fig. 2 exhibits the components and the wiring of the interconnections. The back panel has been opened out, but the resistance boxes are not present. The sockets to which the connecting wires from the resistance boxes are plugged in may be seen at the top.

The inclined top panel (Plate I, Fig. 1) contains the coefficient setting switches and their sign switches. The lower front panel is the operating panel and all the current-adjusting potentiometers and their sign switches, on-off and range switches for currents, the residual set switches with their sign switches and the step selector are located in this. The lights which indicate during the iteration the appropriate current-adjusting potentiometer or the residual switch, can also be seen in the photograph.

The step selector (seen with a black knob) and the feed-back selector (white knob) are respectively situated to the right and left of the galvanometer.

The toggle switch to the right of the galvanometer is for switching in or out of the shunt and the jack connection for an external galvanometer is located on its left.

The sequence selector S is located at the top of the right side panel. All the resistance boxes are fixed to the back panel, through octal sockets.

7. *Programming of the problem and operational procedure*

Table I gives in a summarized form, the various settings of the step-selector, feed-back selector and the sequence-selector

TABLE I

| Operation | Process | Selector Switch Positions | | | Items to be set | Items to be adjusted |
|---|---|---|---|---|---|---|
| | | Sequence | Step | Feed-back | | |
| | *GS Process* | | | | | |
| $S_{7b}$ | Standardisation of the 7th column | 2 | .. | 7 | | $x_7$ (b-current potentiometer) |
| A | Iteration | 1 | 1 to 6 | .. | $a_{ij}$ ($i, j = 1$ to 6), | $x_j$ ($j + 7$) |
| B | Reading $x_j$ after convergence | 2 | .. | 1 to 6 | $-a_{i7}$ ($i = 1$ to 6) $= g_i$ | $a_{i7}$ (read off) ($i = 1$ to 6) |
| | *Matrix Multiplications* | | | | | |
| $S_{7b}$ | Standardisation of the 7th column | 2 | .. | 7 | | $x_7$ (b-current potentiometer) |
| C | Feeding the matrix A and the vector X | 2 | .. | 1 to 6 | $a_{ij}$ ($i, j = 1$ to 6), $a_{i7}$ ($i = 1$ to 6) | $x_j$ ($j + 7$) |
| D | Multiplication and reading off | 3 | 1 to 6 | .. | | $a_{i7}$ (read off) ($i = 1$ to 6) |
| | *RH Process* | | | | | |
| $S_6$ | Standardisation of the 6th column | 5 | .. | .. | | $x_6$ |
| E | Setting -b in the 7th column | 6 | 6 | .. | $b = a_{66}/a_{67}$ | $x_7$ (b-current potentiometer) |
| $S_{71}$ | Setting of unit current in the 7th column | 4 | .. | 7 | | $x_7$ (unit current potentiometer) |
| F | Iteration | 3 | 1 to 5 | .. | $a_{ij}$ ($i, j = 1$ to 5) | $a_{i7}$ ($i = 1$ to 5) |
| G | Feed-back | 4 | .. | 1 to 5 | $-a_{i6} = g_i$ ($i = 1$ to 5) | $x_j$ ($j = 1$ to 5) |
| H | Reading off after convergence | | | | | $a_{i7}$ (read off) ($i = 1$ to 5) |

| | | | | | | |
|---|---|---|---|---|---|---|
| | *Secular Equations* | | | | | |
| $S_{71}$ | Setting unit current in the 7th column | 4 | .. | 7 | | $x_7$ (unit current potentiometer) |
| I | To set $x_r = 1$ | 4 | .. | $r$ | $a_{r7} = 1$ | $x_r$ |
| J | Iteration ($\neq r$) | 4 | 1 to 6 excluding $r$) | | $a_{ij}$ ($i, j = 1$ to 6), $x_j$ | $a_{i7}$ ($i \neq r$) |
| K | Iteration at $r$ | 3 | $r$ | .. | $x_r = 1$ | $x_7$ ($b$-current potentiometer) |
| L | Feed-back | 4 | .. | 1 to 6 | $a_{i7}$, $x_r$ | $x_j$ ($j = 1$ to 6, $j \neq r$, 7) |
| M | Reading of X after convergence | | | | | $a_{i7}$ (read off) |
| | *For Reading off* $\lambda$ | | | | | |
| $S_6$ | Standardisation of the 6th column | 5 | .. | .. | | $x_6$ |
| N | Measuring $\lambda$ | 6 | 6 | .. | | $\lambda = a_{66}/a_{67}$ (read off) |
| | *Choleski's Process* | | | | | |
| $S_{7b}$ | Standardisation of the 7th column | 2 | .. | 7 | | $x_7$ ($b$-current potentiometer) |
| O | Determining L & U (2nd rows) | 1 | 1 to 6 | .. | $a_{i7}$ ($= a_{2j}$) | $x_j$ ($j = 1$) or $a_{ij} = u_{ij}$, $j \neq 1$) |
| P | Measuring L | 2 | .. | 1 | $x_j$ ($j = 1$) | $a_{17}$ (read off) |
| Q | Determining L & U (3rd rows) | 1 | 1 to 6 | .. | $a_{i7}$ ($= a_{2j}$) | $x_j$ ($j = 1, 2$); $a_{ij} = (u_{ij}, j \geqslant 3)$ |
| R | Measuring L | 2 | .. | 1 & 2 | $x_j$ ($j = 1, 2$) | $a_{i7} = (i = 1, 2)$ |

required for the different types of processes, which could be carried out on this model.

(a) *Simultaneous Equations*:

(i) *Gauss-Seidel method. (Operation $S_{7b}$, A and B of Table I)*

To solve $AX = G$, the $6 \times 6$ matrix is first suitably scaled so that the largest coefficient is $\leqslant 1{\cdot}000$. Then the matrix is fed in directly on the coefficient set-switches in the inclined top panel. The $g_i$ values (scaled if necessary, so as to have them all $\leqslant 1{\cdot}000$) are fed in, in the front panel with negative signs. The sequence selector is kept in position 1, for the GS process and the step selector is also kept in the first position. The shunt is introduced for th galvanometer. The computer is now ready for operation.

All the current circuits are put on and the seventh column current potentiometer is adjusted to have a convenient value. If necessary a standard value can be used by comparison with a reference by putting the sequence selector in the 2nd position and the feed-back selector in the 7th position operation $S_{7b}$ of Table I).

The step selector is now moved to the positions 1 to 6, each time adjusting the corresponding $x$-potentiometer for balance in the galvanometer. When the unbalance has become small, the shunt is removed and the cycles are repeated until all the positions of the step-selector show good balance.

With the shunt on, the sequence selector is now turned to the second position and the feed back selector is kept at the first position, and the first residual set-switch is adjusted for balance, the final adjustment being done with the shunt removed. This is repeated at the second and succeeding settings of the feed-back selector (operation B of Table I).

The values indicated by the switches $a_{i7}$ give the solution of the equations to three significant figures.

An example of the solution of the following set of 6 simultaneous equations by the GS method is given in Table II.

$$\begin{bmatrix} 1{\cdot}00 & 0{\cdot}40 & 0{\cdot}20 & 0{\cdot}15 & 0{\cdot}15 & 0{\cdot}10 \\ 0{\cdot}10 & 1{\cdot}00 & 0{\cdot}30 & 0{\cdot}05 & 0{\cdot}06 & 0{\cdot}07 \\ 0{\cdot}10 & 0{\cdot}20 & 1{\cdot}00 & 0{\cdot}04 & 0{\cdot}05 & 0{\cdot}06 \\ 0{\cdot}30 & 0{\cdot}20 & 0{\cdot}40 & 1{\cdot}00 & 0{\cdot}10 & 0{\cdot}07 \\ 0{\cdot}07 & 0{\cdot}08 & 0{\cdot}09 & 0{\cdot}10 & 1{\cdot}00 & 0{\cdot}05 \\ 0{\cdot}04 & 0{\cdot}06 & 0{\cdot}05 & 0{\cdot}03 & 0{\cdot}02 & 1{\cdot}00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0{\cdot}675 \\ 0{\cdot}456 \\ 0{\cdot}347 \\ 0{\cdot}518 \\ 0{\cdot}410 \\ 0{\cdot}456 \end{bmatrix} \quad \ldots \quad (6)$$

TABLE II (a)

Galvanometer deflections in scale divisions for the iterated cycles in the GS process. Trial solution in the first cycle was (1, 0, 0, 0, 0, 0) (Operation A).

| Cycle | Step selector position | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | With Shunt | —F.S. | —17 | —8 | —6 | —15 | —20 |
| 2 | With Shunt | +10 | +3 | +½ | −½ | −1 | ~0 |
| 3 | Without Shunt | −25 | +3 | ~0 | ~0 | −½ | ~0 |
| 4 | Without Shunt | +½ | +½ | ~0 | ~0 | ~0 | ~0 |
| 5 | Without Shunt | ~0 | ~0 | ~0 | ~0 | ~0 | ~0 |

TABLE II (b)

Table showing the current measurement readings. The second number is the deflection of the galvanometer for the setting indicated. (Operation B).

| Feed back selector position | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| | 0·399,+1 | 0·301,+1 | 0·200,+½ | 0·200,+½ | 0·301,−1 | 0·402,−1 |
| | 0·400,−½ | 0·300,−½ | 0·199,−½ | 0·199,−½ | 0·300,+½ | 0·401,+½ |
| Correct values | 0·400 | 0·300 | 0·200 | 0·200 | 0·300 | 0·400 |

(ii) *Matrix multiplication*:

In this case also, after standardization (operation $S_{7b}$), the matrix A and the vector X which are to be multiplied are fed in (operation C). The product can be directly read off (operation D).

(iii) *The Richardson-Hotelling process:*

(*operations* $S_6$, *E*, $S_{71}$, *F*, *G and H of Table I*):

This process can be mechanised for a fifth order equation. The iteration that has to be performed is

$$(A - bI)\, X^{(p)} - G = -bX^{(p+1)} \qquad .. \quad (7)$$

Therefore, when A is given, (A—bI) is first calculated $b$ is a scalar parameter chosen arbitrarily for the given matrix ($b \simeq (\lambda_1+\lambda_n)/2$, where $\lambda_1$ and $\lambda_n$ can be determined quickly by the method to be described in the next sub-section). The $5 \times 5$ matrix $(A - bI)$ is fed in the inclined top panel. The $g_i$ values, with negative signs, are fed in the sixth column, at the top panel.

The 6th column is standardised as mentioned in section 4(c) [operation $S_6$] and then $b$ current potentiometer is adjusted to have the required value, again as in section 4(d) [operation E]. The 7th column is then standardised with unit current potentiometer [operation $S_{71}$]. Then the operations F and G are alternately performed (iteration and feeding back), until the residual set-switches show constant values. These values ($a_{i7}$) give the solution of the set of equations.

(b) *Secular Equations:*

(*operations* $S_{71}$, *I, J, K, L, M,* $S_6$ *and N of Table I*)

(i) *Largest and least eigenvalues.*

Since the analogy to be set up is

$$AX^{(p)} = \lambda X^{(p+1)} \qquad .. \quad (8)$$

the computer can solve 6th order secular equations.

The matrix A is fed in the top panel. With the sequence selector in position 4, and feed-back selector in position 7, the unit current potentiometer is adjusted to standardised 7th column ($S_{71}$). One of the components of the eigenvector, preferably the largest found by trial (say $x_r$) is set to unity by operation I and the corresponding $a_{r7}$ is set to unity.

Now the sequence selector is kept in position 3, and with the step selector in positions 1 to 6 (excluding $r$) the $a_{i7}$ (residual switches $i \neq r$) are adjusted (operation J) and with the step selector in the $r$th position, the ($b$-current potentiometer) is adjusted,

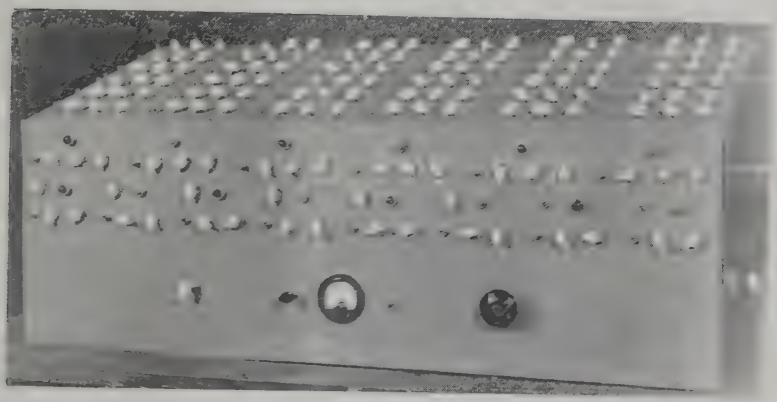

FIG. 1. Front View of Model III

FIG. 2. Back View of Model III

[operation K]. Then the feed-back [operation L] is carried out, adjusting all the $x'_j s$ $(j \neq r, 7)$. These operation J, K, L are then repeated in a cyclic manner until convergence. Then operations M, $S_6$ and N give the values of X and $\lambda_1$.

For determining the least eigenvalue the matrix $(A - \lambda_1 I)$ is fed in the top panel and the above method carried out.

(ii) *Intermediate eigenvalues*:

To obtain these, Richardson's purification process can be applied.

In order to mechanise this process, any arbitrary vector is taken and multiplied by $(A - \lambda_1 I)$ and then the iterated multiplications by A is performed. This will converge to $\lambda_2$. But in this process, the $X_1$ component will tend to reinstate itself; so, we must multiply with $(A - \lambda_1 I)$ from time to time by changing the matrix in the top panel.

In general when we already have $r$ vectors and eigenvalues, we can multiply with $(A - \lambda_1 I)\ (A - \lambda_2 I) \ldots (A - \lambda_r I)$ at regular intervals with a number of iterated multiplications by A, interspaced. This would give the $(r + 1)$th eigenvector and eigenvalue.

The other methods such as the deflation method and the orthogonalisation method, mentioned in Part III, can also be carried out on the computer, but these have not been tested out in detail.

The eigenvalues and vectors of the matrix (9) obtained with the computer is given in Table III. Here $p$ is the total number of iterations (multiplications by A) and for obtaining intermediate eigenvalues the multiplication by $(A - \lambda_1 I) \ldots\ldots (A - \lambda_r I)$ was done once in every $q$ iterations with A. Table IV gives the cosine of the angle between the various vectors $\dfrac{\tilde{X}_k X_l}{\sqrt{X_k \cdot X_k}\sqrt{X_l \cdot X_l}}$ and also the angles between these vectors in degrees. This serves to check the orthogonality of the vectors.

$$\begin{bmatrix} 1{\cdot}000 & 0{\cdot}170 & 0{\cdot}060 & -0{\cdot}070 & -0{\cdot}030 \\ 0{\cdot}170 & 0{\cdot}940 & 0{\cdot}075 & -0{\cdot}200 & -0{\cdot}040 \\ 0{\cdot}060 & 0{\cdot}075 & 0{\cdot}630 & 0{\cdot}020 & -0{\cdot}013 \\ -0{\cdot}070 & -0{\cdot}200 & 0{\cdot}020 & 0{\cdot}470 & 0{\cdot}000 \\ -0{\cdot}030 & -0{\cdot}040 & -0{\cdot}013 & 0{\cdot}000 & 0{\cdot}006 \end{bmatrix} \quad .. \quad (9)$$

TABLE III

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$* |
|---|---|---|---|---|
| 0·971 | −1·000 | 0·105 | 0·055 | 0·031 |
| 1·000 | 0·881 | 0·045 | 0·342 | 0·040 |
| 0·240 | 0·015 | −1·000 | −0·230 | 0·021 |
| −0·341 | −0·280 | −0·220 | 1·000 | 0·023 |
| −0·062 | 0·000 | 0·015 | −0·021 | 1·000 |
| $\lambda_1$=1·19 | $\lambda_2$=0·813 | $\lambda_3$=0·624 | $\lambda_4$=0·401 | $\lambda_5$=0·003 |
| $p$=8; $q$=0. | $p$=15; $q$=5. | $p$=25; $q$=3. | $p$=40; $q$=2. | $p$=10; $q$=0. |

*$X_5$ was obtained by iterated multiplications by $(A - \lambda_1 I)$ and $\lambda_5$ was calculated from $X_5$ by using the last row of the matrix (9).

TABLE IV

$\Sigma\lambda_i = 3{\cdot}031$, $\Sigma a_{ii} = 3{\cdot}046$, $C_{kl} = \text{Cos}\,\theta_{kl}$

| $kl$ | 12 | 13 | 14 | 15 | 23 | 24 | 25 | 34 | 35 | 45 |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_{kl}$ | ·004 | ·013 | ·004 | ·005 | ·014 | ·02 | ·0003 | ·01 | ·005 | ·01 |
| $\theta_{kl}$ | 89°48′ | 89°15′ | 89°48′ | 89°48′ | 89°12′ | 89° | 90° | 89°36′ | 89°48′ | 89°36′ |

## 8. *Other applications*

This computer can also be applied to other problems involving linear equations. Thus it can directly be used for factorisation of a matrix into two triangular matrices, one upper and the other lower. This would help in directly determining the solution of a set of simultaneous equations. This method is due to Choleski (Hartree, [1] 1952).

The method consists in factorising A, as

$$A = LU \quad \ldots\ldots (10)$$

and so

$$AX = LUX = G \quad \ldots\ldots (11)$$

or

$$LY = G \quad \ldots\ldots (12)$$

and

$$UX = Y \quad \ldots\ldots (13)$$

Since L is known, Eqn. (12) can be solved for Y by back substitution and, X can be got again by back substitution in Eqn. (13). This method has been tried out and found to be practicable. The

operations required for Choleski's method are given at the end of Table I ($S_{7b}$, O, P, Q, R).

Incidentally since Det A = (Det L). (Det U) and since Det $L = l_{11}.l_{22}....l_{nn} = 1$ and Det $U = u_{11}.u_{22}....u_{nn}$, the determinant can be quickly evaluated in this way.

Since this method is a direct method, rounding off errors will be cumulative and this would impair the accuracy of the solutions obtained.

This computer can also be used for the inversion of a matrix of order 6, since this only involves the solution of 6 sets of six simultaneous equations (Hartree, 1952).

## 9. *Acknowledgements*

The author wishes to express his deepest sense of gratitude to Prof. G. N. Ramachandran for the helpful guidance and encouragement received during the course of these investigations.

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# The Algebra of Biquadratic Residues

BY

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It was proved by R. Vaidyanathaswamy [2], that the classes $C_1$, $C_2$...., into which the elements of a group G fall with respect to any group of automorphisms of G, combine among themselves by the group operation, so that each element of any class $C_k$ occurs the same number of times among the elements which result on combining each element of $C_i$ with each element of $C_j$ by the group operation. As illustrations of this general principle, he considered (a) the cyclic group of order $p$ (an odd prime) considered as the additive group of the $p$ residue classes (mod $p$) [1]; (b) the algebra of quadratic residues [3].

In another paper which appeared recently [4], he considered the algebra of cubic residues. In this paper which may be considered as continuation of [4], I consider the algebra connected with the biquadratic residues, and obtain, as simple consequences of this algebra, the following known and classical results:

*Theorem* (*A*) The number 2 is a biquadratic residue of an odd prime $p$ if and only if $p$ is of the form (i) $a^2+64d^2$ or (ii) $-1$ (mod 8).

*Theorem* (*B*) The number 3 is a biquadratic residue of an odd prime $p$ if and only if $p$ has any one of the following forms:

(i) $-1$ (mod 12); (ii) $144d^2+a^2$, $a \equiv -1$ or $-5$ (mod 12); (iii) $(12\lambda+9)^2+(12\mu\pm2)^2$.

The forms (ii) are for primes $p$ of the form 1 (mod 8); and (iii) are for $p$'s of the form 5 (mod 8).

It has to be noted that 2 is not a biquadratic residue of primes of the form $\pm 3$ (mod 8) since it is not even a quadratic residue of such primes, and also every quadratic residue of $p$ is also a bi-

quadratic residue of $p$ if $p=-1$ or $-5$ (mod 8). Since for $p=-1$ (mod 8), 2 is a quadratic residue, it is also a biquadratic residue.

2. At the outset we note the following lemmas without proof:

*Lemma* 1. If $p$ is a prime of the form 1 (mod 8), it can be uniquely represented as $(4k-1)^2+16c^2$ and $(p-1)/8=k$ (mod 2). Here and in the next lemma, $k$ is an integer, not necessarily positive.

*Lemma* 2. If $p=5$ (mod 8) is a prime, it can be uniquely represented as $(4k+1)^2+4c^2$ where $c$ is odd and $(p-5)/8=k$ (mod 2).

*The Algebra for* $p=8n+1$: Let $\delta$ be a primitive root of $p$ so that $\delta^{8n}=1$ and $\delta^x \neq 1$ for $x < 8n$.

The $p$ numbers (mod $p$) fall into the following five classes:

$$R_0 = [0]; \quad \pi_0 = \delta^0, \delta^4, \ldots, \delta^{8n-4}$$
$$\pi_1 = \delta^1, \delta^5, \ldots, \delta^{8n-3}$$
$$\pi_2 = \delta^2, \delta^6, \ldots, \delta^{8n-2}$$
$$\pi_3 = \delta^3, \delta^7, \ldots, \delta^{8n-1}$$

By the general principle, stated at the beginning, these five classes combine among themselves by the group operation (addition in this case). Thus we get equations of the following form:

$$R_0 \cdot \pi_i = \pi_i \quad (i = 0, 1, 2, 3)$$
$$\pi^2 = x_1 R_0 + \Sigma a_i \pi_i$$
$$\pi_0 \cdot \pi_1 = x_2 R_0 + \Sigma b_i \pi_i$$
$$\pi_0 \cdot \pi_2 = x_3 R_0 + \Sigma c_i \pi_i$$
$$\pi_0 \cdot \pi_3 = x_4 R_0 + \Sigma d_i \pi_i$$

Here $\pi_i \cdot \pi_j$ stands for adding each element of $\pi_i$ to each element of $\pi_j$. Now since $(-1)$ is a biquadratic residue of $p = 8n+1$, it follows that in each of these classes $\pi_i$, the $2n$ terms occur in $n$ pairs, each pair containing numbers of the form $(r, -r)$. Thus $x_1 = 2n$; $x_2 = x_3 = x_4 = 0$. For the same reason, it also follows that the coeff. of $\pi_i$ in $\pi_j \cdot \pi_k$ = coeff. of $\pi_j$ in $\pi_i \cdot \pi_k$ = coeff. of $\pi_k$ in $\pi_i \cdot \pi_j$.

Thus

$$b_0 = a_1; \quad c_0 = a_2; \quad d_0 = a_3; \quad c_1 = b_2; \quad d_1 = b_3; \quad d_2 = c_3.$$

Also

$$\delta^2 (\pi_0 \cdot \pi_2) = (\delta^2 \pi_0) \cdot (\delta^2 \pi_2) = \pi_2 \cdot \pi_0,$$

so that $c_2 = c_0$; $c_3 = c_1$.

Again

$$\delta^3(\pi_0\cdot\pi_1)=\pi_3\cdot\pi_0, \text{ giving}$$
$$d_3=b_0;\ d_0=b_1;\ d_1=b_2;\ d_2=b_3.$$

Using these relations, the class algebra simplifies as follows:

$$\pi_0^2=2nR_0+a_0\pi_0+a_1\pi_1+a_2\pi_2+a_3\pi_3$$
$$\pi_0\cdot\pi_1=a_1\pi_0+a_3\pi_1+b_2(\pi_2+\pi_3)$$
$$\pi_0\cdot\pi_2=a_2(\pi_0+\pi_2)+b_2(\pi_1+\pi_3)$$
$$\pi_0\cdot\pi_3=a_3\pi_0 b_2(\pi_1+\pi_2)+a_1\pi_3.$$

Next considering the total number of elements on both sides of the equations, we get three new relations:

$$a_0+a_1+a_2+a_3=2n-1;$$
$$a_1+a_3+2b_2=2n;\ a_2+b_2=n.$$

These can be written as follows, taking $a_1$, $b_2$ as the basic unknowns:

$$a_0=3b_2-n-1 \qquad \ldots (1)$$
$$a_2=n-b_2;\ a_3=2n-2b_2-a_1.$$

Since this class algebra is associative, we have $\pi_0^2\cdot\pi_1=\pi_0\cdot(\pi_0\cdot\pi_1)$. Hence comparing in this the coeff. of $\pi_1$, we get, after using equation (1), the relation

$$5b_2^2+a_1^2+2a_1b_2-2na_1-(6n+1)b_2+2n^2=0.$$

Treating $a_1$, $b_2$ as variables, this represents an ellipse whose centre is given by $\xi=\dfrac{4n+1}{8};\ \eta=\dfrac{4n-1}{8}$.

Writing

$$b_2=\xi+B_2;\ a_1=\eta+A_1, \qquad \ldots (2)$$

we get $5B_2^2+A_1^2+2B_2A_1=\dfrac{1}{16}(8n+1)$;

writing $8A_1=\alpha$; $8B_2=\beta$, this becomes

$(\alpha+\beta)^2+4\beta^2=4(8n+1)=4[(4k-1)^2+16c^2]$, by Lemma 1. Noting that $a_1$, $b_2$ are integers we see that $\alpha$, $\beta$ are also integers, though not necessarily positive, we get the following eight possible solutions

$$\alpha+\beta=\pm 2(4k-1);\ \beta=\pm 4c \qquad \ldots (3)$$
$$\alpha+\beta=\pm 8c;\ \beta=\pm(4k-1) \qquad \ldots (4)$$

The solutions (3) are impossible since

$$\alpha+\beta=8(A_1+B_1)=8(a_1+b_1)-8n\equiv 0 \pmod 8,$$

contradicting the first of the equations (3) . Again, among the four solutions in (4), $\beta=-(4k-1)$ is impossible, since

$$\beta=8B_2=8b_2-(4n+1)=-1 \pmod 4.$$

Thus we are left with two alternative solutions

$$\beta=4k-1;\ \alpha+\beta=\pm 8c \qquad .. \quad (5)$$

The ambiguity in the sign of the relation can be resolved only by knowing some more information about the primitive root $\delta$ used. Taking the solution as given by $\beta=4k-1$; $\alpha+\beta=+8c$, we obtain the following scheme for the class algebra.

$$\pi_0^2=2nR_0+(3b_2-n-1)\pi_0+(n-b_2+c)\pi_1$$
$$+(n-b_2)\pi_2+(n-b_2-c)\pi_3.$$
$$\pi_0\cdot\pi_1=(n-b_2+c)\pi_0+(n-b_2-c)\pi_1+b_2(\pi_2+\pi_3)$$
$$\pi_0\cdot\pi_2=(n-b_2)(\pi_0+\pi_2)+b_2(\pi_1+\pi_3)$$
$$\pi_0\cdot\pi_3=(n-b_2-c)\pi_0+b_2(\pi_1+\pi_2)+(n-b_2+c)\pi_3.$$

where $b_2=\dfrac{k+n}{2}$. The scheme corresponding to the other solution is obtained by replacing $c$ by $-c$ in the above.

3. *The Number 2 as a Biquadratic Residue*

We can now prove the following:

*Theorem*: The number 2 is a biquadratic residue of $p=8n+1$ if and only if it is of the form $p=a^2+64d^2$.

*Proof*: At the outset it is evident that 2 is a quadratic residue of $p=8n+1$.

Again it is clear that 2 is a biquadratic residue of $p$ if and only if the coeff. of $\pi_0$ in $\pi_0^2$ is odd,

i.e., $3b_2-n-1\equiv 1 \pmod 2$

or $b_2-n\equiv 0 \pmod 2$.

Now $b_2-n=\dfrac{k+n}{2}-n=\dfrac{k-n}{2}=k(k-1)+c^2$,

using $8n+1=(4k-1)^2+16c^2$

Thus $\frac{k-n}{2} = 0 \pmod 2$ if and only if $c^2$ is 0 mod 2, or $c$ must be even, $c=2d$ (say), so that

$$8n+1=(4k-1)^2+64d^2=a^2+64d^2,\ a \text{ being any odd number.}$$

*Remark*: Since $(2/p)=(-1)^{(p^2-1)/8}$, $p$ being any odd prime, it follows that 2 is not a quadratic residue (and hence not a biquadratic residue) of primes of the form $8n\pm3$, while it is a quadratic of a prime of the form $8n-1$; but then it is also a biquadratic residue, vide [1] Theorem 68. Hence we obtain the theorem (A) above.

4. *The Number 3 as a Biquadratic Residue*

It is easily seen, using the reciprocity law for quadratic residues, that 3 is a quadratic residue of $4n \pm 1$ if and only if $n$ is a multiple of 3; also by a well known result it follows that a quadratic residue of a prime $-1 \pmod 4$ is also a biquadratic residue. Thus we see that 3 is a biquadratic residue of a prime of the form $-1 \pmod{12}$. It remains to investigate for primes of the form $1 \pmod 4$. We first take the form $p = 1 \pmod 8$; the case $p = 5 \pmod 8$ is considered in section (6).

*Theorem* (C). The number 3 is biquadratic residue of a prime $p = 8n + 1$ if and only if $p$ is of the form $a^2 + 144d^2$ where $a = -1$ or $-5 \pmod{12}$.

*Proof*: The coeff. of $\pi_0$ in $\pi_0^3$ must always be in one of the forms $0 \pmod 3$ or $1 \pmod 3$. The latter case will occur only when the same element $a$, in particular 1, is taken from each $\pi_0$ in $\pi_0^3$, and then $3a$ will be an element in $\pi_0^3$ belonging to $\pi_0$. Thus in particular 3 is a biquadratic residue of $p$ if and only if, E, the coeff. of $\pi_0$ in $\pi_0^3$ is $1 \pmod 3$. Now,

$$E=2n+(3b_2-n-1)^2+(n-b_2+c)^2+(n-b_2)^2+(n-b_2-c)^2 \quad .. \quad (6)$$

$$=2n+(n+1)^2+2c^2 \pmod 3$$

$$=n^2+n+1+2c^2 \pmod 3 \quad .. \quad (7)$$

The expression E in (6) is symmetric in $b$ and $-b$, and so is the same whichever of the two solutions is taken.

Now, $8n+1=(4k-1)^2+16c^2$

$$n=2c^2+2k^2-k \quad .. \quad (8)$$

Using the results that for any integer $\beta$, we have

$$\beta^3 \equiv -2\beta \pmod 3; \quad \beta^4 \equiv -2\beta^2 \pmod 3$$

we get $n^2 \equiv -2c^2 - k^2 - k^2c^2 - kc^2 + 2k$ .. (9)

Hence from (7),

$$\begin{aligned} E &= n^2 + n + 1 + 2c^2 \pmod 3 \\ &= 2c^2 + k^2 + k + 1 - kc^2 - k^2c^2, \text{ using (8) and (9)} \\ &= k(k+1) + 1 - c^2(k+2)(k-1) \\ &= k(k+1) + 1 - c^2(k-1)^2 \pmod 3 \end{aligned}$$

Now let $k$ and $c$ take all possible values, $0, \pm 1 \pmod 3$, we can easily show that $E = 1 \bmod 3$, in the cases $k \equiv 0, c \equiv 0$, or $k \equiv -1$, $c \equiv 0 \pmod 3$ and $E = 0 \pmod 3$ in all other cases. Hence the theorem follows.

5. *The Algebra for* $p = 8n + 5$,

Let $\delta$ be a primitive root and we form, as in the previous cases, the classes

$$R_0 = [0]$$
$$\pi_i = \delta^i, \ \delta^{i+4}, \ \ldots, \ \delta^{i+8n}, \ (i = 0, 1, 2, 3)$$

These classes are mutually exclusive and together cover all the integers mod $p$.

(A). Since $(-1)$ is a quadratic residue, but not a biquadratic residue of $p = 8n + 5$, the elements of $\pi_2$ are the same as those in $\pi_0$ with opposite signs. Similarly, the elements of $\pi_3$ are the same as those of $\pi_1$ with opposite signs.

It follows from the general principle, mentioned at the beginning, that the classes form an algebra of the form

$$\begin{aligned} \pi_0^2 &= \Sigma a_r \pi_r \\ \pi_0 \cdot \pi_1 &= \Sigma b_r \pi_r \\ \pi_0 \cdot \pi_2 &= \Sigma c_r \pi_r \\ \pi_0 \cdot \pi_3 &= \Sigma d_r \cdot \pi_r. \qquad (r = 0, 1, 2, 3) \qquad .. \quad (10) \end{aligned}$$

In view of the property (A) mentioned above, we see that

coeff. of $\pi_2$ in $\pi_0 \cdot \pi_3$ = coeff. of $\pi_1$ in $\pi_0^2$, and in general

coeff. of $\pi_i$ in $\pi_r \cdot \pi_s$ = coeff. of $\pi_{r+2}$ in $\pi_{i+2} \cdot \pi_s$

These relations give

$$d_2 = a_1; \ b_2 = a_3; \ c_3 = b_0; \ d_3 = b_1; \ d_0 = c_1; \ c_2 = a_0$$

Also, $\delta^2(\pi_0 \cdot \pi_2) = \pi_2 \cdot \pi_0$.

Hence

$$c_0\pi_2+c_1\pi_3+a_0\pi_0+b_0\pi_1 = c_0\pi_0+c_1\pi_1+a_0\pi_2+b_0\pi_3 \quad .. \quad (11)$$

Thus $c_0=a_0$; $c_1=b_0$ on equating coeff. of like classes. Similarly

$$\delta^3(\pi_0\cdot\pi_1) = \pi_3\cdot\pi_0$$

and proceeding as before we get $b_0=b_1$; $a_3=d_1$; $b_3=a_1$ .. (12)

Using (10), (11) and (12) the algebra can be shown by the following scheme.

$$\pi_0^2 = \Sigma a_i\pi_i,\ i=0,\ 1,\ 2,\ 3$$
$$\pi_0\cdot\pi_1 = b_0(\pi_0+\pi_1)+a_3\pi_2+a_1\pi_3$$
$$\pi_0\cdot\pi_2 = a_0(\pi_0+\pi_2)+b_0(\pi_1+\pi_3)$$
$$\pi_0\cdot\pi_3 = b_0(\pi_0+\pi_3)+a_3\pi_1+a_1\pi_2$$

Equating the total number of terms in the classes on both sides, we get,

$$a_0+a_1+a_2+a_3=2n+1$$
$$a_1+a_3+2b_0=2n+1$$
$$a_0+b_0=n.$$

Taking $a_0$ and $a_1$ as the basic variables these equations can be written as

$$b_0=n-a_0;\ a_3=1+2a_0-a_1$$
$$a_2=2n-3a_0.$$

Using the associative property of the group operation, we have $\pi_0\cdot(\pi_0\cdot\pi_1)=(\pi_0\cdot\pi_0)\cdot\pi_1$ and equating the coeff. of $\pi_0$ in the expression for the two sides we get

$$5a_0^2-2a_0a_1+a_1^2+(2-4n)a_0-a_1+(n^2-n)=0.$$

Writing $a_0=A_0+\dfrac{4n-1}{8}$; $a_1=A_1+\dfrac{4n+3}{8}$

this becomes $5A_0^2+A_1^2-2A_0A_1=(8n+5)/16.$ .. (13)

Writing $X=8A_0$; $Y=8A_1$ so that X and Y are integers (not necessarily positive) since $a_0$, $a_1$ are integers $\geqslant 0$, we get

$$4X^2+(X-Y)^2=4(8n+5)=4(a^2+b^2)$$

where say $a$ is odd, $b$ even.

Thus the possible solutions of this Diaphantine eqn. are

$$X=\pm a;\ X-Y=\pm 2b \quad .. \quad (14)$$

or $$X=\pm b;\ X-Y=\pm 2a \quad .. \quad (15)$$

But since from (13), $X=4k+1 \pmod 8$

$$Y=-(4n+3) \pmod 8$$

the solution (15) is impossible, since then $X = \pm b$, even. Again of the four possible solutions in (14) X will be equal to $a$ or $-a$, whichever is of the form $4k+1$ (mod 8) and the only ambiguity remaining is the solution

$$X - Y = \pm 2b$$

The actual determination of the sign depends upon the primitive root $\delta$ chosen. Thus writing $a=4k+1$; $b=2c$ we have

$$p=8n+5=(4k+1)^2+4c^2 \qquad .. \quad (17)$$

(this must be a unique representation for $p$), where $c$ will necessarily be odd, $(=2l+1)$ say; the possible solutions will be

$$a_0=k(k+1)+\frac{c^2-1}{4}=k(k+1)+l(l+1)$$

$$a_1=(n+k\mp c+1)/2=\frac{n+k}{2}\mp l.$$

and the final scheme of the algebra is

$$\pi_0^2=a_0\pi_0+a_1\pi_1+(2n-3a_0)\pi_2+(1+2a_0-a_1)\pi_3.$$
$$\pi_0\cdot\pi_1=(n-a_0)(\pi_0+\pi_1)+(1+2a_0-a_1)\pi_2+a_1\pi_3$$
$$\pi_0\cdot\pi_2=a_0(\pi_0+\pi_2)+(n-a_0)(\pi_1+\pi_3)+(2n+1)R_0$$
$$\pi_0\cdot\pi_3=(n-a_0)(\pi_0+\pi_3)+(1+2a_0-a_1)\pi_1+a_1\pi_2.$$

6. *The Number 3 as a Biquadratic Residue of $8n+5$.*

The coeff. of $\pi_0$ in $\pi_0^3$ is

$$E=a_0^2+a_1(n-a_0)+(2n-3a_0)a_0+(1+2a_0-a_1)(n-a_0)$$
$$\equiv 2a_0^2+(n-1)a_0+n \pmod 3$$

which is independent of the ambiguous value of $a_1$.

Now from (17), $a_0=k(k+1)+l(l+1)$

$$n=(2k+1)k+2l(l+1)$$

Taking all possible values for $k$ and $l$ (viz.) 0, $\pm 1$ (mod 3 we can show that $E \equiv 1 \pmod 3$ only when $k \equiv -1$; $l \equiv 0$ or $\equiv -1$ (mod 3), so that in these cases only 3 is a biquadratic residue of $p$. In other cases E will be 0 (mod 3) and so 3 will be a biquadratic nonresidue of $p$.

When $E \equiv 1 \pmod 3$, we can put $k=3\lambda+2$; $l=3\mu$ or $3\mu+2$ so that

$$p=(12\lambda+9)^2+4(6\mu+1)^2$$
or $$(12\lambda+9)^2+4(6\mu+5)^2$$

Thus if 3 is a biquadratic residue of $p \equiv 5 \pmod 8$ it must have the form $(12\lambda+9)^2+(12\mu\pm2)^2$. This combined with theorem (C) gives theorem (B).

7. In the general case of a prime $p$ of the form $\lambda k+1$ we can study the algebra of $k$th power residues, by forming the classes $R_0$, $\pi_i$ given by, $R_0 \equiv 0$, and $\delta$ being a primitive root of $p$,

$$\pi_i = \delta^i,\ \delta^{i+k},\ \ldots,\ \delta^{i+(\lambda-1)k},\ i = 0,\ 1,\ \ldots,\ (k-1).$$

The integer $l$ will be a $k$th power residue of $p$ if and only if the coeff. of $\pi_0$ in $\pi_0^l$ is 1 (mod $l$); and a nonresidue if the same is of the form 0 (mod $l$). A detailed study of this case will be taken up in a separate paper.

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# Liquid-Liquid Extraction Studies with Propylene Glycol

BY

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ABSTRACT

In connection with the study of extraction of propylene glycol from its aqueous solution, the present work was undertaken to evaluate the performance of solvents, iso-Amyl alcohol, n-Butanol, methyl isobutyl ketone, isoAmyl acetate, n-Butyl acetate and ethyl acetate in terms of the ternary mutual solubility and tie line data at 30°C. The performance of these solvents indicate that they could be used as selective solvents in extracting propylene glycol. However, the equilibrium distribution results indicate that the ratio of propylene glycol concentration in the aqueous phase to that in the above solvents is greater than unity.

## *Introduction*

The technique of liquid-liquid extraction is gaining in importance in recent years because of its applicability to many liquid separation processes of commercial significance where other vaporisation methods are either uneconomical or actually impossible on account of the similar vapour pressures, azeotrope-forming properties, non-volatile nature or heat sensitiveness. Hence, the importance of study of phase equilibrium and solubility data of ternary liquid systems needs to be much emphasized. For assessment of the relative usefulness of various solvents for extraction, it is necessary to have equilibrium and phase distribution data for the liquid systems.

In connection with the study of dehydration of alcohol by extractive distillation Ramanujam (1958) has suggested the use of polyhydric alcohols such as ethylene glycol, propylene glycol or glycerol as entrainers. The bottoms in these cases is an aqueous solution of the diol or triol. Also, in the manufacture of propylene glycol from propylene oxide the product obtained is a very dilute solution of propylene glycol. Since large amounts of water have to be evaporated off in order to recover pure glycol these methods become costlier.

Laddha and Smith (1948) has suggested the application of liquid-liquid extraction technique to recover ethylene glycol from aqueous solutions and considered the possibility of n-hexyl alcohol and n-amyl alcohol as solvents. The performance of a number of other solvents such as n-butanol, isobutanol, M.E.K., methyl acetate, ethyl formate, cyclohexanol, triethylamine, isobutyraldehyde etc., have been evaluated by many research workers (1956, 1959).

In the present investigation with propylene glycol, the performance of the solvents, n-butanol, pri-isoamyl alcohol, isoamyl acetate, n-butyl acetate, ethyl aceate and methyl isobutyl ketone has been evaluated. The tie line data have been correlated according to the method of Othmer and Tobias (1942).

*Materials used and Experimental procedure*:

1. Propylene glycol: Propylene glycol supplied by British Drug Houses Ltd., was used after its purification by distillation under vacuum. It had a density of $d_4^{30}$ 1·0293.

2. n-Butanol: n-Butyl alcohol supplied by E. Merck Darmstadt was fractionated and the fraction boiling between 117·7 and 118°C was used. It had a density of $d_4^{30}$ 0·8021.

3. Pri-isoamyl alcohol: Primary isoamyl alcohol A.R. grade supplied by Chempure Ltd., Calcutta, was fractionated and the fraction boiling at 132°C was collected and used. It had a density $d_4^{30}$ 0·8018.

4. Methyl isobutyl ketone: Laboratory reagent grade methyl isobutyl ketone supplied by British Drug Houses Ltd., was fractionated and the fraction distilling at 116°C was used. It had a density of $d_4^{30}$ 0·7922..

5. Iso amyl acetate: Isoamyl acetate supplied by Leonard J. Hymes, New York was purified by distillation and the fraction distilling at 142°C was collected and used. It had a density of $d_4^{30}$ 0·8634.

6. Butyl acetate: n-Butyl acetate supplied by E. Merck Darmstadt was fractionated and the distillate boiling at 126°C was collected and used. It had a density of $d_4^{30}$ 0·8727.

7. Ethyl acetate: Supplied by E. Merck, Darmstadt was fractionally distilled and the fraction distilling at 77°C was used. It has a density of $d_4^{30}$ 0·8881.

8. Water: Laboratory supply of distilled water was used.

Experimental Procedure:

Known mixtures by weight of propylene glycol and solvent were taken and titrated with the third component until turbidity appeared. These were placed in the constant temperature bath maintained at $30 \pm 0{\cdot}1°C$. The titration was carried in several steps in order that the mixture could be returned frequently to the constant temperature bath to ensure maintenance of 30°C. The end point was indicated sharply by a permanent turbidity resulting from the addition of the last drop of the diluent to the previously clear solution. The density of the resulting solution was determined at 30°C for each experiment and the density composition curve was prepared for each ternary liquid system. Similar experiments were carried out for the diluent-rich phase taking known mixtures of diluent and solute and titrating them against the solvent. Check experiments were carried out over the entire range of the solubility curve.

The tie lines were located by preparing mixtures of the three components which could separate into two layers at equilibrium. After vigorous agitation the mixtures were placed in the constant temperature bath at $30 \pm 0{\cdot}1°C$ for a sufficiently long time and allowed to come to equilibrium. Frequently the mixture was shaken and finally allowed to separate into two layers. The individual phases were then separated and then weighed. The density of each of the layers was also determined at the same temperature. The composition of each phase was then evaluated from the density composition curves. The tie lines obtained were checked with the weight ratio of weights of the conjugate phases.

TABLE I

Equilibrium Data for the System Propylene Glycol-Water-Isoamyl alcohol at 30°C

| Solubility Data | | | Weight Percentage | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | Amyl alcohol | Prop. glycol | Water | Amyl alcohol |
| — | 9·78 | 90·22 | — | 97·61 | 2·39 |
| 8·32 | 13·53 | 78·15 | 4·94 | 92·54 | 2·52 |
| 10·90 | 15·40 | 73·40 | 7·48 | 89·98 | 2·54 |
| 15·01 | 17·49 | 67·50 | 12·32 | 85·09 | 2·59 |
| 19·38 | 21·71 | 58·91 | 19·50 | 77·68 | 2·92 |
| 22·60 | 25·63 | 51·77 | 26·68 | 69·67 | 3·65 |
| 23·66 | 27·06 | 49·28 | 32·68 | 59·29 | 8·03 |
| 31·07 | 24·06 | 44·87 | — | — | — |

Tie Line Data

| Solvent Phase Composition Wt % | | | Water Phase Composition Wt % | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | Amyl alcohol | Prop. glycol | Water | Amyl alcohol |
| 2·15 | 10·7 | 87·15 | 6·75 | 90·73 | 2·52 |
| 5·55 | 12·1 | 82·35 | 14·25 | 83·15 | 2·60 |
| 8·83 | 14·22 | 76·95 | 20·6 | 76·42 | 2·98 |
| 13·8 | 17·05 | 69·15 | 26·7 | 69·55 | 3·75 |
| 18·7 | 20·8 | 60·50 | 30·3 | 64·45 | 5·25 |

TABLE II

Equilibrium Data for the System: Prop. Glycol-Water-n-Butanol at 30°C

| Solubility Data | | | Weight Percentage | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | n-Butanol | Prop. glycol | Water | n-Butanol |
| — | 10·23 | 79·77 | 15·91 | 58·47 | 25·62 |
| 4·69 | 14·17 | 71·14 | 16·17 | 66·96 | 16·87 |
| 7·86 | 28·61 | 63·53 | — | 93·05 | 6·95 |
| 11·42 | 36·05 | 52·53 | 4·50 | 88·48 | 7·02 |
| 13·23 | 42·06 | 44·71 | 9·87 | 82·66 | 7·47 |
| 14·42 | 46·70 | 38·88 | 11·42 | 80·47 | 8·11 |
| 15·50 | 54·38 | 30·12 | 13·74 | 76·71 | 9·55 |
| 15·52 | 56·13 | 28·35 | 15·09 | 73·54 | 11·37 |

Tie Line Data

| Solvent Phase Composition Wt % | | | Water Phase Composition Wt % | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | n-Butanol | Prop. glycol. | Water | n-Butanol |
| 1·2 | 21·0 | 77·8 | 3·5 | 89·5 | 7·0 |
| 3·9 | 23·6 | 72·5 | 6·3 | 86·6 | 7·1 |
| 5·4 | 25·3 | 69·3 | 8·6 | 84·0 | 7·4 |
| 8·1 | 28·9 | 63·0 | 12·0 | 79·8 | 8·2 |
| 10·3 | 33·7 | 56·0 | 14·4 | 75·6 | 10·0 |

TABLE III

Equilibrium Data for the System Prop. Glycol-Water-Methyl Isobutyl Ketone at 30°C

| Solubility Data | | | Weight Percentage | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | M.I.B.K. | Prop. glycol | Water | M.I.B.K. |
| — | 1·80 | 98·20 | — | — | — |
| 10·79 | 3·22 | 85·99 | 57·47 | 29·80 | 12·73 |
| 20·48 | 5·27 | 74·25 | — | 98·44 | 1·56 |
| 27·76 | 6·76 | 65·48 | 7·86 | 90·37 | 1·77 |
| 32·69 | 8·17 | 59·14 | 25·68 | 72·02 | 2·30 |
| 41·02 | 10·87 | 48·11 | 40·00 | 56·54 | 3·46 |
| 47·12 | 13·27 | 39·61 | 45·12 | 50·29 | 4·59 |
| 55·58 | 18·16 | 26·26 | 54·73 | 36·12 | 9·15 |

Tie Line Data

| Solvent Phase Composition Wt % | | | Water Phase Composition Wt % | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | M.I.B.K. | Prop. glycol | Water | M.I.B.K. |
| 0·25 | 1·65 | 98·1 | 12·5 | 85·61 | 1·89 |
| 1·5 | 2·00 | 96·5 | 28·25 | 69·35 | 2·4 |
| 3·85 | 2·15 | 94·0 | 45·0 | 50·4 | 4·6 |
| 6·25 | 2·65 | 91·0 | 53·0 | 39·05 | 7·95 |
| 8·51 | 3·0 | 88·49 | 56·75 | 31·35 | 11·9 |

TABLE IV

Equilibrium Data for the System: Prop. Glycol-Water-Isoamyl acetate at 30°C

| Solubility Data | | | Weight Percentage | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | Isoamyl acetate | Prop. glycol | Water | Isoamyl acetate |
| — | 1·07 | 98·93 | — | 99·77 | 0·23 |
| 6·55 | 0·99 | 92·46 | 8·11 | 91·62 | 0·26 |
| 12·81 | 1·02 | 86·17 | 12·00 | 87·73 | 0·27 |
| 22·81 | 1·03 | 76·16 | 23·43 | 76·24 | 0·33 |
| 29·02 | 1·10 | 69·88 | 32·59 | 66·94 | 0·47 |
| 42·03 | 1·52 | 56·45 | 46·70 | 52·44 | 0·56 |
| 55·27 | 2·31 | 42·42 | 54·01 | 44·41 | 1·58 |
| 61·45 | 2·73 | 35·82 | 64·92 | 32·12 | 2·96 |
| 69·62 | 4·43 | 25·95 | 71·81 | 22·46 | 5·73 |
| 72·29 | 5·93 | 21·78 | 75·61 | 13·23 | 11·16 |

Tie Line Data

| Solvent Phase Composition Wt % | | | Water Phase Composition Wt % | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | Isoamyl acetate | Prop. glycol | Water | Isoamyl acetate |
| 0·4 | 1·1 | 98·5 | 9·6 | 90·1 | 0·3 |
| 1·2 | 1·4 | 97·4 | 26·7 | 72·8 | 0·5 |
| 2·6 | 1·0 | 96·4 | 47·0 | 52·1 | 0·9 |
| 4·0 | 0·9 | 95·1 | 65·0 | 32·2 | 2·8 |
| 5·2 | 1·0 | 93·8 | 74·0 | 18·9 | 7·1 |

TABLE V

Equilibrium Data for the System Prop. Glycol-Water-n-Butyl acetate at 30°C

| Solubility Data | | | Weight Percentage | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | n-Butyl acetate | Prop. glycol | Water | n-Butyl acetate |
| — | 1·14 | 98·86 | 67·39 | 13·35 | 19·26 |
| 5·90 | 1·43 | 92·67 | — | 99·39 | 0·61 |
| 10·75 | 1·68 | 87·57 | 8·98 | 90·29 | 0·73 |
| 18·01 | 2·26 | 79·73 | 18·89 | 80·26 | 0·85 |
| 27·50 | 3·07 | 69·43 | 32·88 | 66·08 | 1·04 |
| 31·39 | 3·47 | 65·14 | 49·67 | 47·07 | 3·26 |
| 44·80 | 4·89 | 50·31 | 57·97 | 38·32 | 3·71 |
| 57·04 | 7·11 | 35·85 | 65·36 | 27·23 | 7·41 |

Tie Line Data

| Solvent Phase Composition Wt % | | | Water Phase Composition Wt % | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | n-Butyl acetate | Prop. glycol | Water | n-Butyl acetate |
| 0·5 | 1·2 | 98·3 | 26·3 | 72·7 | 1·0 |
| 2·0 | 1·2 | 96·8 | 46·0 | 52·3 | 1·7 |
| 3·9 | 1·3 | 94·8 | 60·3 | 35·1 | 4·6 |
| 7·4 | 1·6 | 91·0 | 68·0 | 18·3 | 13·7 |

TABLE VI

Equilibrium Data for the System Prop. Glycol-Water-Ethyl actate at 30°C

| Solubility Data | | | Weight Percentage | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | Ethyl acetate | Prop. glycol | Water | Ethyl acetate |
| — | 2·71 | 97·29 | — | 93·21 | 6·79 |
| 7·48 | 4·65 | 87·87 | 10·33 | 82·07 | 7·60 |
| 15·82 | 8·37 | 75·81 | 24·25 | 66·36 | 9·39 |
| 22·96 | 12·71 | 64·33 | 35·18 | 51·52 | 13·30 |
| 27·67 | 15·64 | 56·69 | 39·72 | 41·50 | 18·78 |
| 32·93 | 20·01 | 47·06 | 40·50 | 32·65 | 26·85 |
| 37·76 | 24·91 | 37·33 | — | — | — |

Tie Line Data

| Solvent Phase Composition Wt % | | | Water Phase Composition Wt % | | |
|---|---|---|---|---|---|
| Prop. glycol | Water | Ethyl acetate | Prop. glycol | Water | Ethyl acetate |
| 1·6 | 3·0 | 95·4 | 10·4 | 81·8 | 7·8 |
| 2·7 | 3·1 | 94·2 | 19·2 | 72·41 | 8·39 |
| 3·9 | 3·6 | 92·5 | 25·5 | 64·96 | 9·54 |
| 5·3 | 4·0 | 90·7 | 34·2 | 53·1 | 12·7 |
| 6·8 | 4·3 | 88·9 | 38·6 | 44·7 | 16·7 |

TABLE VII

Dielectric Constants of Solvents (1955)

| No. | Substance | Dielectric constant | Temperature |
|---|---|---|---|
| 1. | n-Butylalcohol .. | 17·1 | 25°C |
| 2. | Pri-isoamyl alcohol .. | 14·7 | 25°C |
| 3. | Methyl isobutyl ketone .. | 13·11 | 20°C |
| 4. | Iso amyl acetate .. | 4·63 | 30°C |
| 5. | n-Butyl acetate .. | 5·01 | 20°C |
| 6. | Ethyl acetate .. | 6·02 | 25°C |

TABLE VIII

Plait Points for the Systems

| No. | System | Plait Points Wt % | | |
|---|---|---|---|---|
| | | Prop. glycol | Water | Solvent |
| 1. | Prop. Glycol-Water-Isoamyl alcohol .. | 28·89 | 37·52 | 33·59 |
| 2. | Prop. Glycol-Water-n-Butanol .. | 14·39 | 46·30 | 39·31 |
| 3. | Prop. Glycol-Water-M.I.B.K. .. | 17·25 | 4·47 | 78·28 |
| 4. | Prop. Glycol-Water-Isoamyl acetate .. | 15·00 | 1·01 | 83·99 |
| 5. | Prop. Glycol-Water-n-Butyl acetate .. | 12·8 | 1·8 | 85·4 |
| 6. | Prop. Glycol-Water-Ethyl acetate .. | 12·2 | 6·8 | 81·0 |

*Results and Correlations*:

The ternary solubility and phase equilibrium data for the systems at 30°C are presented in Tables I to VI. The triangular diagrams for the systems showing the binodal curves are plotted in Figures 1 to 6. Equilibium distribution diagram for various solvents is shown in Figure 7. In order to determine the relative

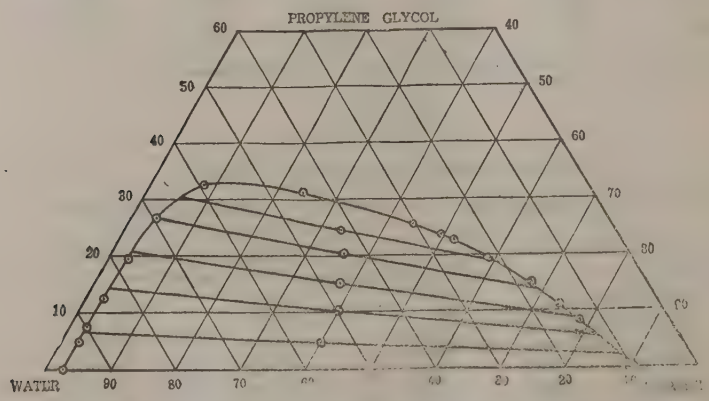


Fig 1. Propylene Glycol-water—Isoamyl Alcohol System

degree of selectivity of the various solvents, equilibrium selectivity diagram for these systems have been plotted in Figures 8, 9 and 10. These diagrams are made by plotting the solute content

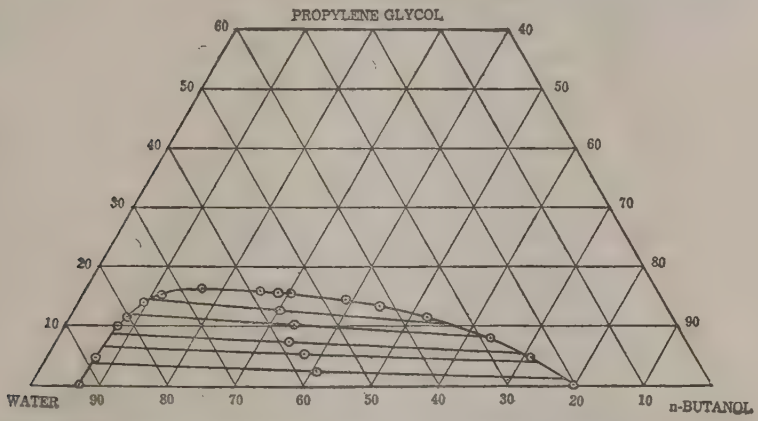


FIG. 2. Propylene Glycol-water—*n*-Butanol System

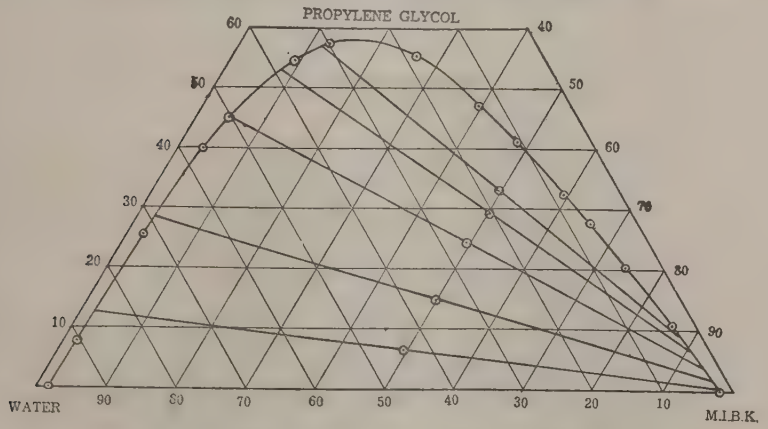


FIG. 3. Propylene Glycol-water—Methyl Isobutyl Ketone System

in aqueous phase against the solute content in solvent phase on solvent free basis. The selectivity of the solvents for extraction of propylene glycol increases with decrease in the value of dielectric constants of chemically similar groups of solvents such as alcohols, esters etc. A comparison of the ternary phase

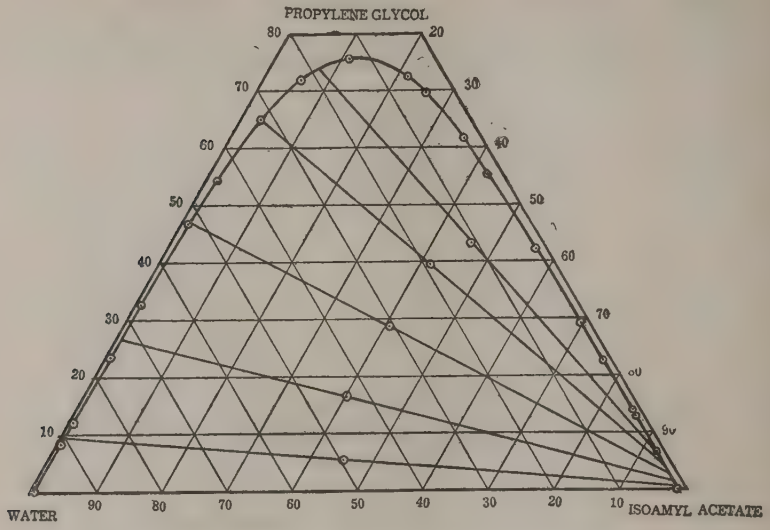


FIG. 4. Propylene Glycol-water—Isoamyl Acetate System

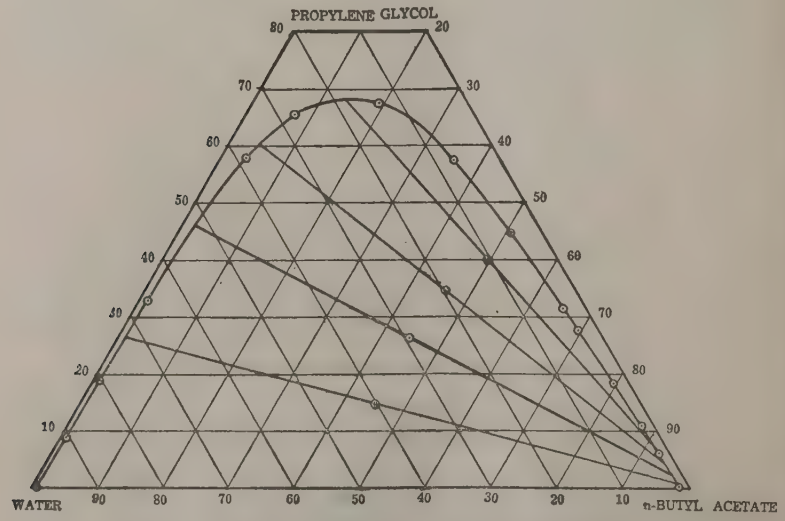


FIG. 5. Propylene Glycol-water—*n*-Butyl Acetate System

diagrams (Figures 1 & 6) reveals the fact that the area of heterogeneity under the binodal curve increase with increase in molecular weight and with decrease in dielectric constant in chemically similar groups of solvents. Similar behaviour have been observed in the case of extraction of ethylene glycol from aqueous solutions (1959).

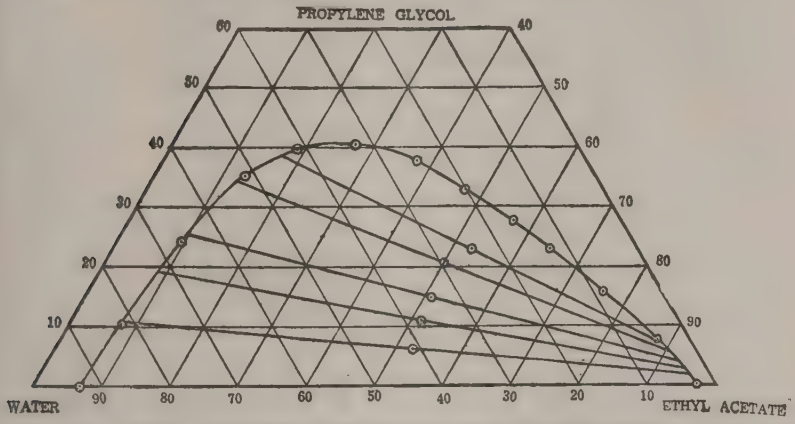


Fig. 6. Propylene Glycol-water—Ethyl Acetate System

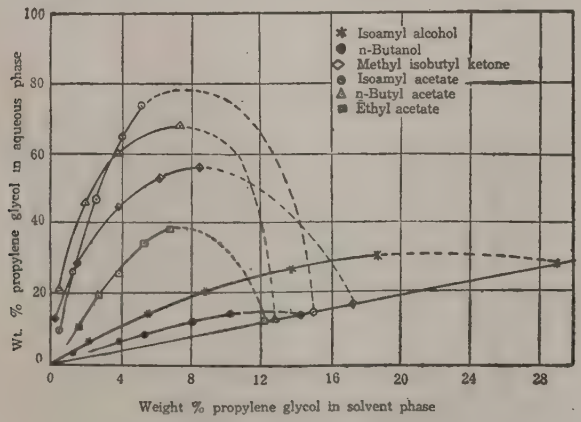


Fig. 7. Equilibrium Distribution Diagram

The ternary phase distribution data for these systems were correlated according to the methods suggested by Othmer and Tobias, Bachman, Campbell and Hand. Othmer and Tobias

correlation is shown in Figure 11. The plait points have been determined by methods described by Treybal (1946). An illustra-

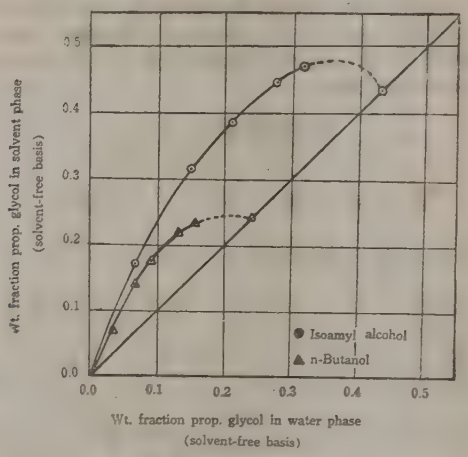


FIG. 8. Selectivity Diagram

tion of the method has been shown in Figure 12. Compositions of plait points by this method are given in Table VIII.

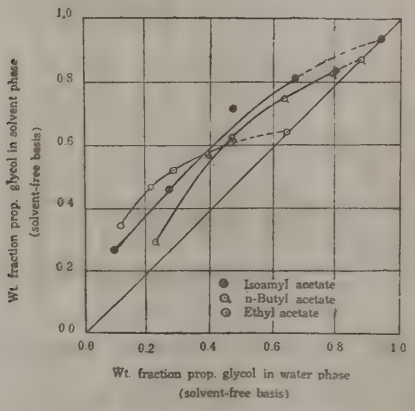


FIG. 9. Selectivity Diagram

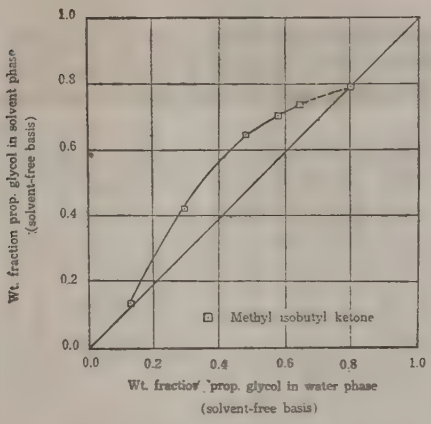


FIG 10. Selectivity Diagram

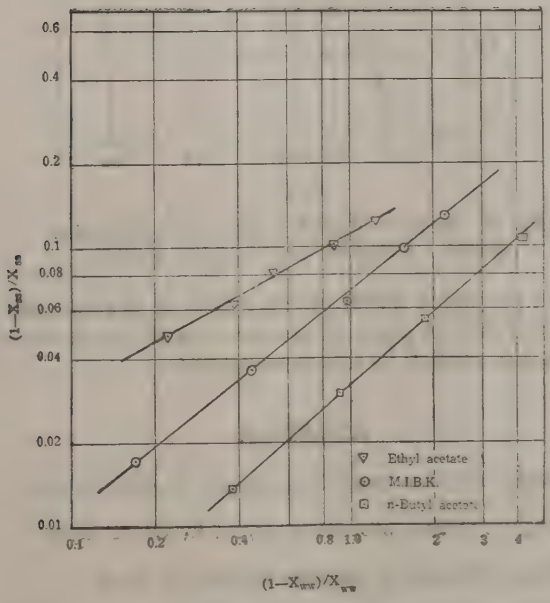


FIG. 11. Othmer & Tobias correlation of the tie line Data

*Conclusion:*

A comparison of the equilibrium results indicate that alcohols are more selective than ketone and esters with reference to extraction of propylene glycol from aqueous solutions. However,

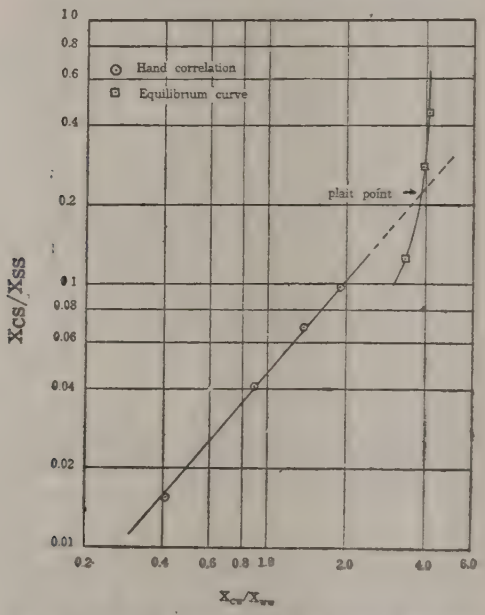


FIG. 12. Treybal's Method of Plait Point Estimation
System: Propylene Glycol-Water—M.I.B.K.

the ratio of propylene glycol concentration in the aqueous phase to that in the solvent phase is greater than unity, in the case of all the solvents studied in this investigation.

## NOTATION

$X_{cs}$ Weight fraction propylene glycol in solvent layer

$X_{cw}$ Weight fraction propylene glycol in water layer

$X_{ss}$ Weight fraction of solvent in solvent layer

$X_{ww}$ Weight fraction of water in water layer

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# Paracompactness Conditions in Subsets

BY

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ABSTRACT

Two types of paracompactness conditions are introduced on a subset of a topological space, stronger than paracompactness under relative topology, and their properties studied with examples. One of them is used to generalise a product theorem concerning paracompact spaces.

1. The notion of paracompactness in a subset of a topological space is usually considered with respect to the relative topology. In the case of compact Hausdorff spaces, the compact subsets are identical with the closed sets of the space. This situation does not obtain for paracompact spaces because of the notion of local finiteness involved in their definition. This fact and some of the results of Michael in [2] lead us to consider two types of paracompactness conditions in a subset of a topological space, which are stronger than paracompactness with respect to relative topology. We call them strong paracompactness and $\sigma$-strong paracompactness and define these by varying the domain of local finiteness in two (equivalent) definitions of a paracompact space. Strongly paracompact subsets of a Hausdorff paracompact space are shown to be identical with the closed subsets of the space. Defining a completely paracompact space as one in which every subset is $\sigma$-strongly paracompact, we prove that the product of a completely paracompact space and a metric space is paracompact. This generalises the result of Michael, [2], that the product of a paracompact perfectly normal space and a metric space is paracompact and perfectly normal.

2. Let S be a topological space and A a subset of it. A collection of subsets of S whose union is A is called a *covering* of A. A covering $\mathbf{U}$ of A is called *S-locally finite* if every point of S has a neighbourhood which meets only a finite number of sets of $\mathbf{U}$. By a *refinement* of $\mathbf{U}$ is meant a covering $\mathbf{V}$ which is such that for every $V \in \mathbf{V}$ there exists a $U \in \mathbf{U}$ so that $U \supset V$. The space S is

paracompact when every open covering of S has an S-locally finite open refinement. Michael [2] has shown that a regular space is paracompact if and only if every open covering **U** of S has a $\sigma$-S-locally finite (not necessarily open) refinement, i.e., a refinement $\mathbf{V} = \overset{\infty}{\underset{i=1}{\mathrm{U}}} \mathbf{V}_i$ where each $\mathbf{V}_i$ is S-locally finite.

A subset A of S is a paracompact subset when it is paracompact under the relative topology, i.e., if every relatively open covering of A has an A-locally finite refinement consisting of relatively open subsets of A. It is known that closed subsets and F$\sigma$-subsets (countable union of closed subsets, see [2]) of a Hausdorff paracompact space are paracompact subsets.

We define strongly paracompact and $\sigma$-strongly paracompact subsets as follows:

A subset A of S is *strongly paracompact* if every relatively open covering of A has an S-locally finite relatively open refinement.

A subset A of S is *$\sigma$-strongly paracompact* if every relatively open covering of A has a $\sigma$-S-locally finite relatively open refinement i.e., a relatively open refinement $\mathbf{V} = \overset{\infty}{\underset{i=1}{\mathrm{U}}} \mathbf{V}_i$ where each $\mathbf{V}_i$ is an S-locally finite collection of subsets of S.

It is clear that strong paracompactness implies $\sigma$-strong paracompactness in any space S, while in regular spaces, $\sigma$-strong paracompactness implies paracompactness with respect to relative topology. Examples are given below in art. 4 to show that the reverse implications are not true.

3. Let us consider first the strongly paracompact subsets of S. It is easy to verify that

(3.1) *Any closed subset of a paracompact space is strongly paracompact.*

The converse is true when the space is Hausdorff.

(3.2) *A strongly paracompact subset of a Hausdorff space is necessarily closed.*

*Proof*: Let A be a strongly paracompact subset of a Hausdorff space S and let $x$ be a point of S outside A. We show that $x$ cannot be an accumulation point of A. For every point $y$ of A there is an open neighbourhood $\mathrm{U}(y)$ of $y$ which is disjoint with an open neighbourhood $\mathrm{U}_y(x)$ of $x$. Consider the covering of A

formed by the intersections with it of $\{U(y) \mid y \in A\}$. Since A is strongly paracompact there exists a relatively open S-locally finite refinement V of this covering. Now let $N(x)$ be the neighbourhood of $x$ which meets only a finite number of members of V, and let $\{U(y_i)\}$, $i = 1, 2, \ldots, n$, be the sets which contain these finite number of members of V. Then the neighbourhood $N(x) \cap \Pi [U_{y_i}(x)]$ is an open neighbourhood of $x$ which cannot meet A. Thus A is closed.

(3.3) It follows from (3.1) and (3.2) that if S is a Hausdorff paracompact space, the family of its closed sets is identical with the family of its strongly paracompact subsets.

The method of proof of (3.2) can be employed to show further that any two disjoint strongly paracompact subsets of Hausdorff space can be enclosed in disjoint subsets of S, which amounts to showing that any two disjoint closed sets in a Hausdorff paracompact space can be separated by disjoint open sets, leading to the well-known fact that a Hausdorff paracompact space is normal.

4. Let us next consider $\sigma$-strongly paracompact subsets of S. While proving that an F$\sigma$-subset of a paracompact Hausdorff space is paracompact, Michael has actually shown that it is $\sigma$-strongly paracompact, (*vide* remark at the end of art. 5 in Michael [2]). A subset A is called a generalised F$\sigma$-subset if every open set containing A also contains an F$\sigma$-subset containing A. Such a set is also $\sigma$-strongly paracompact. This is a consequence of the following result which can be easily verified.

(4.1) *If a subset A of S is such that every open set G containing A also contains a $\sigma$-strongly paracompact set, then A is $\sigma$-strongly paracompact.*

We proceed to give two examples concerning strongly paracompact and $\sigma$-strongly paracompact subsets.

*Example* 1. Let S be an uncountable set in which one point $p$ is distinguished. It is topologised by prescribing that a subset of it is open if (i) it does not contain $p$, or (ii) its complement is finite. Then S is a hereditarily paracompact Hausdorff space which is not perfectly normal. Any subset which contains $p$ is closed and so strongly paracompact by (3.1). Any infinite set not containing $p$ is *not* strongly paracompact (there cannot be 'local finiteness' at $p$) but is paracompact under relative topology (which is discrete).

*Example* 2. Half-open interval space. Let S be the set of all real numbers with the half-open interval topology, i.e., a base is the family of all half-open intervals $[a, b) = \{x | a \leqslant x < b\}$, $a, b$ being real. This space is also hereditarily paracompact Hausdorff and it is perfectly normal. But every subset of it is $\sigma$-strongly paracompact. This is a consequence of the fact that every subspace of S is regular Lindelöf (*vide* Kelley, [1]). Since S is not discrete, every subset of it is not strongly paracompact, by (3.3), so that there exist $\sigma$-strongly paracompact subsets which are not strongly paracompact.

5. The notion of $\sigma$-strong paracompactness is useful in connection with the problem of the product of a paracompact space and a metric space. It is not yet known whether this product should always be paracompact. But Michael [2] has shown that the product of a paracompact perfectly normal space and a metric space is paracompact and perfectly normal. We prove a similar result here. We start with the following definition:

A space is *completely paracompact* if every subset of it is $\sigma$-strongly paracompact.

It is clear that a completely paracompact space is hereditarily paracompact. That the reverse implication is not true is evident from Example 1 above. Also it is easy to prove that:

(5.1) *A space is completely paracompact if and only if every open subset of it is $\sigma$-strongly paracompact.*

The half-open interval space of Example 2 above is an example of a completely paracompact space. This space is, however, perfectly normal. It would be interesting to know an example of completely paracompact space which is not perfectly normal.

THEOREM. *The Cartesian product of a regular completely paracompact space and a metric space is paracompact.*

*Proof*: We adopt Michael's method of proof, [2]. Let S be a regular completely paracompact space and M a metric space. By the Smirnov-Nagata condition for metrisability of a space, (*vide* Kelley, [1]) there exists for M a $\sigma$-M-locally finite open basis $\mathbf{V} = \bigcup_{i=1}^{\infty} \mathbf{V}_i$, where each $\mathbf{V}_i$ is M-locally finite. We shall write $\mathbf{V} = \{V_\alpha\}_{\alpha \in \Lambda}$ To prove that $S \times M$ is paracompact, we shall show that every covering $\mathbf{U}$ of it has a $\sigma$-$(S \times M)$-locally finite refinement, since $S \times M$ is clearly regular.

Now for each $\alpha \in \Lambda$, we choose a collection $\{R_\gamma\}_{\gamma \in \Gamma(\alpha)}$ of open sets of S such that $\{R_\gamma \times V_\alpha\}_{\gamma \in \Gamma(\alpha), \alpha \in \Lambda}$ is an open refinement of **U**. If we write $W_\alpha = U_{\gamma \in \Gamma(\alpha)}$ then $\{R_\gamma\}$ is an open covering of the (open) subset $W_\alpha$ of S, and since $W_\alpha$ is $\sigma$-strongly paracompact, $\{R_\gamma\}$ has an open $\sigma$-S-locally finite refinement $\{T_\beta\}_{\beta \in B(\alpha)}$. Now consider the family of sets $\mathbf{W} = \{T_\beta \times V_\alpha\}_{\beta \in B(\alpha), \alpha \in \Lambda}$. It is an open refinement of **U**. And since $\{T_\beta\}$ is a $\sigma$-S-locally finite, and $\{V_\alpha\}$ is $\sigma$-M-locally finite $\{T_\beta \times V_\alpha\}$ is $\sigma$-$(S \times M)$-locally finite. Hence the theorem.

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# A Contribution to the Embryology of Michelia Champaca

BY

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ABSTRACT

The tapetum in the microsporangium of *Michelia champaca* is of parietal origin. The tapetal cells become binucleate and function in a secretory manner.

Cytokinesis in pollen mother cells occurs by the simultaneous method and the four microspores of a tetrad are grouped in an isobilateral manner.

The ovule is bitegmic. A well developed parietal layer is present.

The embryo sac develops according to the monosporic eight nucleate type.

The endosperm is cellular from the beginning.

The embryo follows the Onagrad type of development.

The seed coat is formed by the outer integument which is differentiated into an outer fleshy layer and an inner hard layer. Oil cells occur singly in the fleshy layer. Endosperm in the mature seed contains large quantities of fatty reserves. Only one layer of nucellus persists in the seed.

## *Introduction*

Recent work on the comparative morphology of the Ranales, started by Professor Bailey and his colleagues, indicates the importance of this order in studies relating to the origin and diversification of the dicotyledons. The necessity for collecting data from various botanical disciplines has been rightly stressed. It is desirable to study the embryology of the ranalean members in more intensity to supplement the informations.

As treated by Dandy (Hutchinson, 1959) the Magnoliaceae consists of ten genera distributed in Southeast North and Central America, West Indies, Venezula, Brazil, East Asia and Malay Archipelago. The embryological literature on the family is rather scanty. The information available upto 1930 has already been

summarized by Schnarf (1931). Maneval (1914) described the gatetophytes of *Magnolia* and *Liriodendron*. Studies on the method of cytokinesis in the pollen mother cells of *Magnolia* were made by Guignard (1898), Andrews (1901) and Farr (1918). Apart from these, discussion on the method of cytokinesis in *Magnolia* are included in the works of Schürhoff (1926) and Farr (1916). The seed structure of *Magnolia* was studied by Asa Gray (1857) and Takhtajan (1948, 1959). Earle (1938) published an account of the ovule, embryo, sac, endosperm and embryo of *Magnolia grandiflora*.

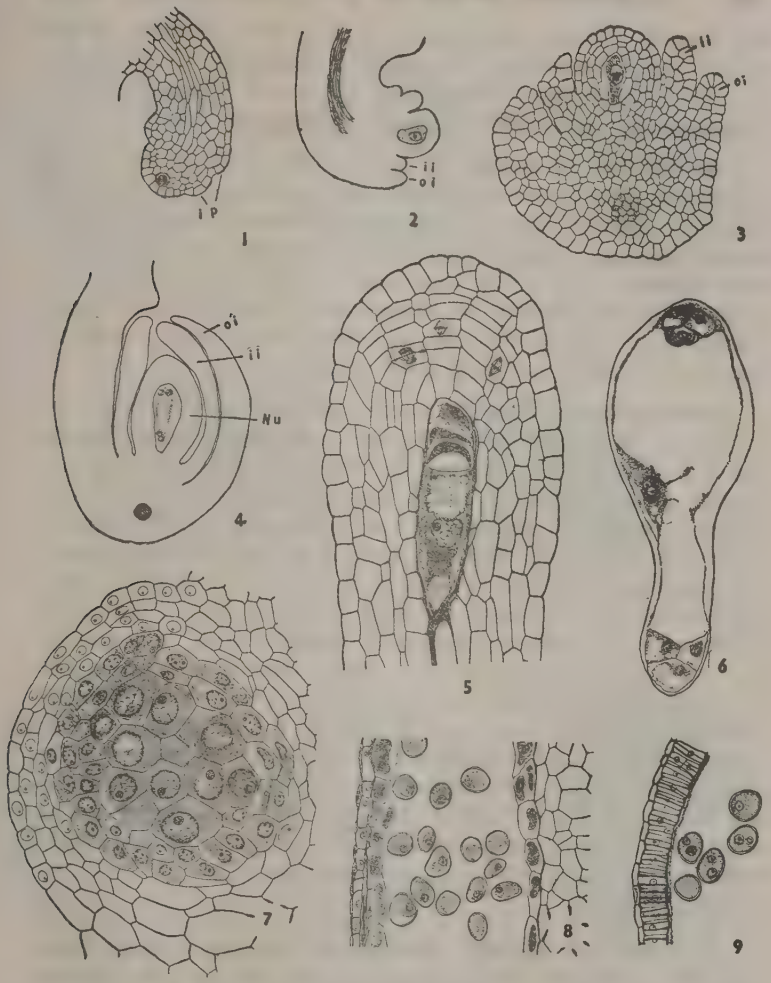
## *Observations*

*Microsporogenesis*:

The archesporium is of hypodermal origin. The division of the archesporial cells results in two layers—the outer one being the parietal layer, the inner one forms the spore mother cells. The parietal layer divides in periclinal planes resulting in three wall layers, besides the epidermis (Text-Fig. 7). The hypodermal layer develops into the endothecium (Text Fig. 9). The innermost layer becomes differentiated as the tapetum (Text-Fig. 7). The middle layer shows tendicies to divide further, resulting in two daughter layers. The tapetum ultimately becomes irregularly two-layered (Text-Fig. 7).

The endothecial cells develop fibrous thickenings at about the time of pollen formation (Text-Fig. 9). The middle layers get crushed at this stage. The tapetal cells become binucleate at the time of meiosis in the pollen mother cells (Text-Fig. 8) and their behaviour conforms to the secretory type.

*Cytokinesis in pollen mother cells*: Prior to the onset of meiosis in the pollen mother cells the nucleus begins to enlarge and a transluscent substance having affinity for coal tar dyes appears between the cell wall and the cytoplasm (Text-Fig. 10). At the corners of the cells, this matter appears separated from the cell wall. During early prophase, the cytoplasm of the pollen mother cell remains uniformly dense (Text-Fig. 10). After the first meiotic division, the daughter nuclei are organised at two poles of the mother cell (Text-Fig. 11). The spindle shows secondary increase in diameter. Accompanying these changes the cytoplasm is indented in the centre by a slowly growing furrow (Text-Fig. 11). Following this, the transluscent matter enters




TEXT-FIGS. 1-9.

Diagrams illustrating megasporogenesis and microsporogenesis. 1. Longisection through an ovule primordium. X 160. 2. Slightly later stage of ovule development. X 250. 3. Vertical section through nucellus of an young ovule showing megaspore mother cell. X 320. 4. Longisection through ovule at four nucleate embryo sac stage. X 125. 5. Vertical section through nucellus showing a megaspore tetrad. X 400. 6. Mature embryo sac. X 400. 7. Transection through young anther lobe showing microspore mother cells. X. 400. 8. Portion of a vertical section through an anther lobe at microspore stage. X 160. 9. Similar section as in Fig 3 at pollen grain stage. X 160. (*oi*—outer integument; *ii*—inner integument).

the furrow. During this process certain outer regions of the spindle get obliterated in the region of furrowing. However, no cell plate was seen forming at this stage (Text-Fig. 11). Subsequently, the spindle of meiosis I disappears.

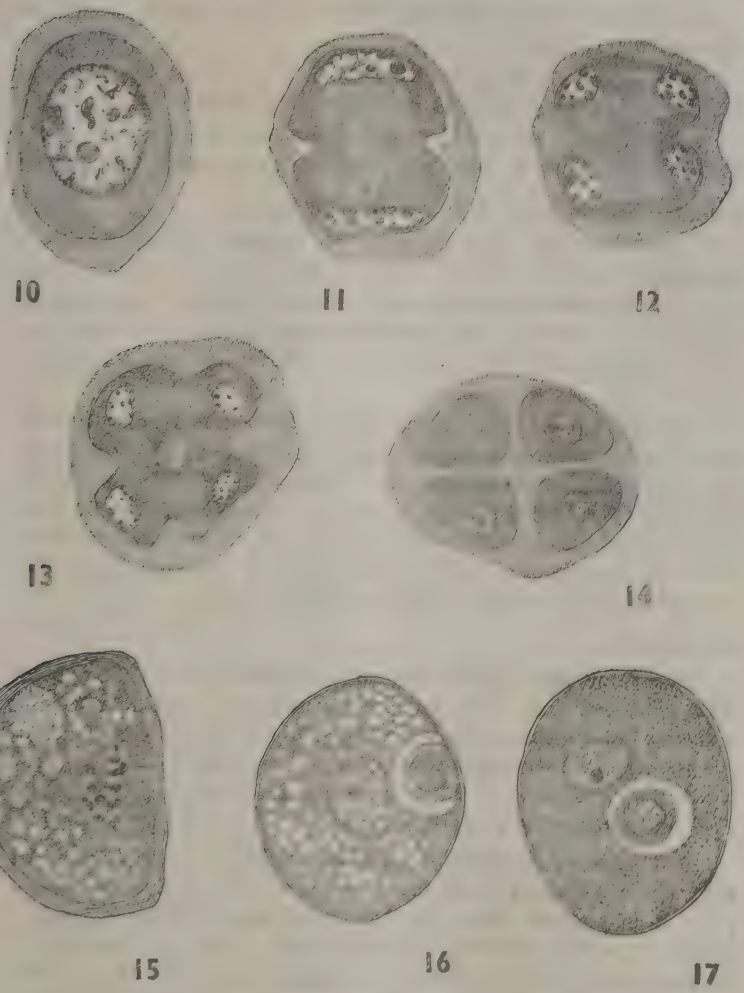
At the time of meiosis II, two spindles, both of which are oriented at right angles to the first one, are formed. At telophase II another annular furrow arises perpendicular to the one already initiated during meiosis I. As a result, the resulting four nuclei are disposed in an isobilateral manner. Due to the centripetal advance of the furrows the original mother cell becomes partitioned into four compartments, each compartment with a microspore nucleus and associated cytoplasm.

After separation from the tetrad configuration, the microspore develops a thick wall and its nucleus undergoes mitosis towards one side of the pollen grain (Text-Fig. 15). The shape of the spindle is noteworthy. It has a tapering inner end and a blunt wallward end. At metaphase, the asymmetry of the spindle reaches its maximum. The cytoplasm at this stage shows numerous vacuoles. After the completion of the division, a small generative cell is cut off towards the wall (Text-Fig. 16). The nucleus of the vegetative cell is considerably larger in size. The generative cell is clearly separated by a highly tenuous membrane that is organized by the phragmoplast. However, the membrane soon disappears and the space can be seen as a line of demarcation (Text-Fig. 16). The generative cell soon looses contact with the spore wall and lies free wihtin the cytoplasm of the vegetative cell (Text-Fig. 17). Thus at shedding stage, the pollen grains are two-celled and the generative cell appears to be completely embedded in the cytoplasm of the vegetative cell.

*Ovule and Nucellus*:

The ovule of *Michelia champaca* is initiated as a papillate projection into the core of which procambium differentiates acropetally in continuation of the strands in the placentum. The distal end of the projection corresponds to the incipient nucellus. Soon a hypodermal archesporial cell becomes differentiated in this region. The outer and inner integuments are originated almost simultaneously (Text-Fig. 1). The integuments arise by divisions in the subepidermal cells lying at the base of the nucellus.

During its development the ovule shows a distinct curvature indicating the onset of anatropus condition. The curvature seems




Text—Figs. 10-17.
Stages in microspore formation. X 1250.

to be initiated by the unilateral growth of the ovule primordium (Text-Figs. 1, 2, 14).

The archesporial cell soon cuts off a parietal cell. The parietal cell undergoes divisions in both anticlinal and periclinal planes; thus organising a well developed parietal tissue (Text-Fig. 5). As

a consequence, the megaspore mother cell gets deeply buried in the nucellus (Text-Figs. 3, 5).

The micropyle is formed by the inner as well as the outer integuments. The vascular strand in the funicle differentiates upto the base of the nucellus and branches are given off at the chalaza. The strands supplying the integument reach upto the tip.

*Megasporogenesis and Female Gametophyte*:

The megaspore enlarges prior to division. After meiosis a linear tetrad of megaspores is formed (Fig. 5). The chalazal measpore enlarges and germinates urther while the other three degenerate. The nucleus of the functional megaspore by three successive divisions, gives rise to an eight-nucleate embryo sac (Text-Fig. 6).

At the two-nucleate stage the embryo sac is considerably enlarged. A central vacuole becomes initiated at the two-nucleate stage and attains conspicuousness when the embryo sac is four-nucleate. The egg apparatus consists of two synergids with basal vacuoles and an egg with apical vacuolation. The three antipodal cells are organised at the chalazal end and the polar nuclei lie juxtaposed near the egg apparatus (Text-Fig. 6).

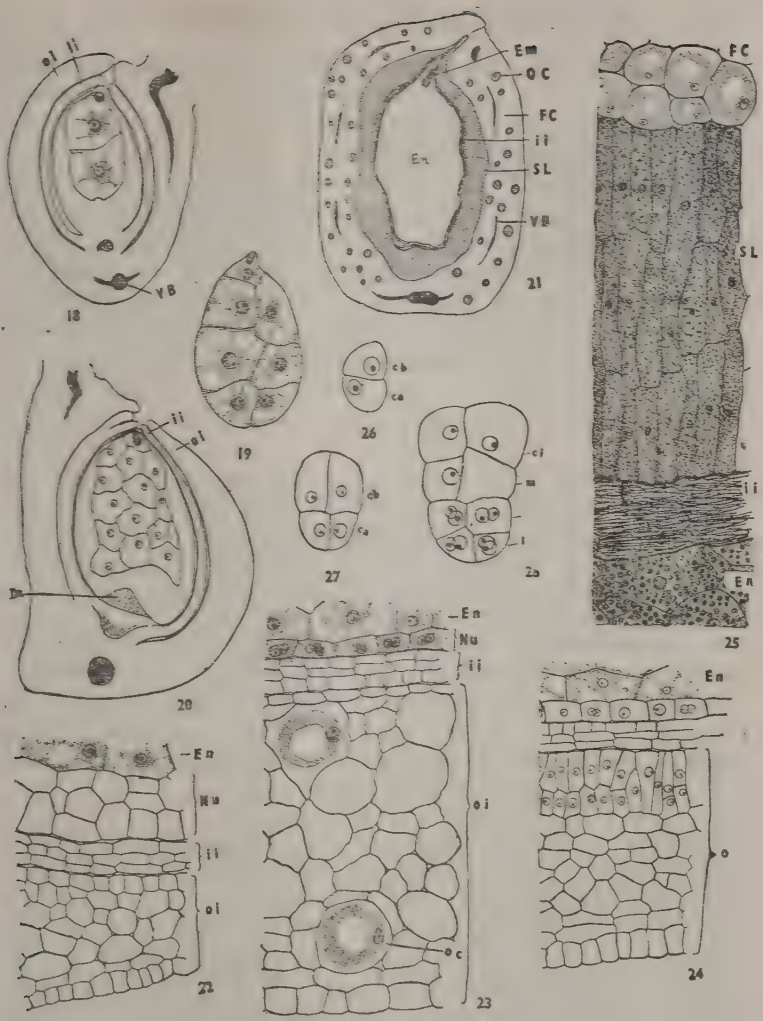
*Endosperm*:

The polar nuclei fuse at the time of fertilization. The pollen tube, after arriving at the apex of the nucellus, makes its way in between the nucellar cells lying above the embryo sac and enters the sac near the egg apparatus. After completion of double fertilization, the remains of the pollen tube are seen in the nucellus. The antipodals soon get degenerated.

The primary endosperm nucleus divides in a plane at right angles to the long axis of the embryo sac and a transverse cell plate is laid down (Text-Fig. 18). The next division occurs by a vertical wall in both the daughter cells. Later divisions are in varying planes and a small mass of large vacuolated cells is produced (Text-Figs. 19, 20). At later stages of seed development the endosperm forms a compact mass of cells rich in reserve food (Text-Figs. 21, 25). Thus the endosperm belongs to the *ab initio* cellular type.

*Embryo*:

The zygote divides only after a considerable mass of endosperm cells is formed. The first division in the zygote is followed




TEXT—FIGS. 18-28.

Diagrams illustrating development of endosperm, embryo and seed. 18. Longisection of ovule showing two-celled endosperm. X. 54. 19. Endosperm at a slightly later stage than in fig. 18. X 54. 20. Longisection through ovule showing cellular endosperm. X 54. 21. Longisection through mature seed. X. 5. 22-24. Portion of the integuments and the nucellus at young stages of seed development. X 213. 25. Section through stony layer of seed coat. X. 213. 26-28. Stages in embryo development. X 533.

*oi*—outer integument; *ii*—inner integument; *Nu*—nucellus; *En*—endosperm; *oc*—oil cell; *SL*—stony layer; *FC*—fleshy layer; *Em*—embryo; *VB*—vascular bundle; *Dn*—degenerated nucellar cells.

by the laying down of a transverse wall (Text-Fig. 26). The basal cell *cb* as well as the apical cell *ca* divide by vertical walls (Text-Fig. 27). Subsequently, another vertical wall at right angles to the first one is formed in the apical cell, resulting in the organization of the quadrants. The quadrant cells divide by transverse walls to engender the octants, which are disposed in two tiers *l* and *l'*. The two juxtaposed cells derived from *cb* undergo transverse divisions. At the third cell generation there are eight cells disposed in three tiers. Thus the embryo development belongs to the Onagrad Type (Johansen, 1950). The mature embryo is fully differentiated into the primary axis and two cotyledons.

*Seed* :

At the time of fertilization, the outer integument consists of a homogeneous mass of parenchyma cells (Text-Fig. 22). The number of cell layers composing the outer integument varies from the chalazal to the micropylar end. The inner integument consists of three layers of cells and is considerably thin.

The developing endosperm absorbs the nucellar layers successively until a single layer is left (Text-Figs. 22-24). However, the persisting nucellus at the chalazal end is more than one-layered and above the termination point of the chalazal vascular strand a dark-staining mass of degenerated nucellar cells is found at late stages of seed development (Text-Fig. 21).

The inner integument gets crushed and its remains are seen as a papery layer in the mature seed. The outer integument develops two zones—an outer fleshy and an inner stony (Text-Fig. 25). During the development of the seed the stony zone arises from cells towards the inner epidermis. These cells elongate radially and divide forming upto three or four layers. Wall thickening does not occur in these cells but hardening is brought about by vacuolar incrustations. It is to be mentioned that the nucleus in these cells remains morphologically healthy even after hardening. The fleshy layer contains isolated ethereal oil cells dispersed among thin-walled parenchyma. The endosperm in the mature seed is massive and contains polyhedral cells with large numbers of oil globules and fatty material (Text-Fig. 25).

The mature seeds hang down by thin threads from the placenta after splitting of the carpel wall. The main constituents of

the threads are the primary xylem elements of the funicle together with the associated parenchymatous tissue.

## *Discussion*

***Anther and Microsporogenesis*:**

In *Magnolia virginiana* (Maneval, 1914) three or four wall layers are cut off in the microsporangium. In *Michelia champaca* also, the number of wall layers is identical. The tapetum is reported to be consisting of two or three layers of cells in *Magnolia virginiana* and *Michelia* is similar even though the two-layered condition is predominant. In both these genera, the tapetal cells become binucleate and are of the secretory type. Well developed endothecium and the two-celled shedding stage of pollen are likewise common features.

Maneval (1914) observed that the tapetal cells in *Magnolia* are of sporogenous origin. Swamy (1949) contradicted this by pointing out that in *Magnolia soulangeana, M. stellata, Michelia fuscata* and *Liriodendron tulipifera* the tapetum is parietal in origin. In *Liriodendron*, according to Maneval (1914) the tapetal nuclei wander into the locule of the anther. This feature should be regarded as an exception in view of the binucleate tapetum being the usual condition rported for other genera of the Magnoliaceae including *Michelia champaca.*

The method of cytokinesis in *Michelia champaca* resembles that in *Magnolia* (Farr, 1913; Andrew, 1901). In both these genera construction furrows are initiated at the close of meiosis I and their activity is revived after meiosis II, so that a simultaneous formation of the four cells is achieved. Guignard (1898) observed in *Magnolia* that there is a secondary increase in the spindle of meiosis I. *Michelia champaca* exhibits similar developments during cytokinesis. Isobilateral configuration of the microspores at the tetrad stage, is common to both the genera. It is obvious that isobilateral tetrads are formed by simultaneous method of cytokinesis in Magnoliaceae and Annonaceae (Johano, 1935; Periasamy and Swamy, 1959). The notable deviation, however, is to be found in the first furrowing which occurs after meiosis I, thus tending to isolate two cells which further subdivide to form four, in the Magnoliaceae. Periasamy and Swamy (1959) observe that in the Annonaceae the secondary spindles differ from the spindles of meiosis I and II in that they are appearances brought about by cytoplasmic striations. A similar observation in *Miche-*

*lia* indicates that the spindles formed between the daughter nuclei are of a different nature.

*Ovule*:

The ovules of *Michelia champaca* and *Magnolia* are anatropous and bitegmic. Well developed parietal tissue occurs in both the genera. The embryo sac development follows the monosporic eight nucleate type in *Michelia* as well as in *Magnolia*. The embryo sac elongates at both ends at the four nucleate stage. In *Michelia* the polar nuclei do not fuse before fertilization whereas in *Magnolia* they fuse prior to the entry of pollen tube. The structure of the ovule and female gametophyte of *Michelia fastuosa* (Swamy, unpublished observations) are identical with those of *Michelia champaca* (present work).

*Endosperm*:

Nuclear endosperm was described in *Magnolia grandiflora* by Earle (1938). But, Swamy (1949) has shown that in *Magnolia virginiana, M. soulangeana, M. stellata, Michelia fuscata* and *Liriodendron tulipifera* the endosperm is cellular from the beginning. *Michelia champaca* falls in line with Swamy's report.

*Embryo*:

The four celled proembryo of *Michelia* is erected by vertical walls in both apical and basal cells. *Magnolia* is similar. The derivatives of the basal cell in both these genera form a massive suspensor. The embryo development in *Michelia* as well as *Magnolia* follows the Myosurus variation, Onagrad Type (Johansen, 1950). The embryo is small compared to the volume of endosperm in the mature seed.

*Seed*:

The structure of the seed coat of *Magnolia* as recently illustrated by Takhtajan (1959) shows a stony layer situated inner to the outer fleshy layer in which ethereal oil cells are dispersed. The inner integument is reduced to a papery structure in the mature seed. The cells of the hard layer are shown with thickened walls. However, as pointed by Swamy (1949) the hardness of the concerned cells is due to vacuolar incrustrations in contrast to the condition in the Annonaceae wherein the hardness is due to lignified secondary walls. In *Michelia champaca* wall thickening does not occur in the hard layer. The hardness is due to vacuolar incrustations as pointed by Swamy (1949). Oil cells in the fleshy

layer occur singly in *Magnolia* and *Michelia* as contrasted to *Degeneria* (Swamy, 1949) where they are disposed in small groups. The endosperm contains fatty reserves in large quantities in *Magnolia* and *Michelia*.

One layer of nucellus persists in the seed of *Michelia* as in Magnolia. The cells of this layer show, often, multinucleate condition.

The cumulative information on the embryological features of *Michelia* is in conformity with the characters that have been described for the Magnoliaceae *sensu strictu*.

*Acknowledgement*

I am deeply indebted to Professor B. G. L. Swamy, for guidance and constant encouragement.

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# Studies on Cercospora Leaf Spot of Groundnuts (Arachis hypogaea)

## I. *Effect of the environmental factors on disease incidence and on survival of the pathogen* *

BY

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ABSTRACT

The 'tikka' disease or the leaf spot of groundnut, caused by *Cercospora personata* (B. & C.) Ell. & Ev., is common in the rainfed crop of South India. A loss in yield due to a premature leaf fall is reported.

The disease manifests itself as black or brown spots with a 'halo' round them with clusters of conidia produced on the under-surface of the leaves.

All cultivated strains of groundnuts are susceptible, though the early maturing strain TMV 2 is found to be the most susceptible. The spread of the disease is effected by the resting mycelia which survive in the soil on the fallen leaves and debris and attack the lower leaves of the rainfed crop when suitable predisposing factors like high relative humidity, prolonged low temperatures and dew prevail.

### *Introduction*

The leaf spot caused by *Cercospora* sp. forms the most destructive disease of the growing groundnut plant and is of wide distribution within the tropical and sub-tropical areas of the world wherever groundnut is cultivated. It was reported from North America (Berkeley, 1875), Java (Raciborski, 1898), West Indies (Bancroft, 1910) and India (Butler, 1914). More recent reports are from the Phillippines, Italy, Japan, South Africa, Australia and China.

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Groundnut is essentially a dryland crop in South India. There are mainly two crops a year, the rainfed crop from July to November-December being the more common which depends on the North East Monsoon. *Cercospora personata* leaf spot is generally widespread in this rainfed crop. It first appears as minute spots on the lower leaves of seedlings when the crop is about a month old. Appearing sometime in August-September, the intensity of the disease gradually increases upto October-November and in years when the environmental conditions are particularly conducive for its spread, it develops into an epidemic. Under these conditions, heavy leaf-fall takes place resulting in reduced yield and in addition, the fallen leaves form the debris on which other fungi like *Sclerotium rolfsii* grow.

In view of the importance of this crop in the economy of this country, studies on the nutritional requirements of the fungus *Cercospora personata* along with the physiology of its relationship with the host was undertaken. The following is an account of some general and preliminary studies made on the symptomatology of the disease and on the environmental conditions favouring the survival and pathogenicity of the fungus.

## *Material and Methods*

*Isolation and culture of the fungus*:

Single spore isolates of the fungus were made from fresh infected material as described earlier (Shanta, 1956). The cultures were maintained in a sporulating condition by selective subculturing of conidia once a fortnight on Czapeck's ($NaNO_3$ : 2·0g., $KH_2PO_4$ : 1·0g., $MnSO_4 \cdot 7H_2O$ : 0·5g., KCl : 0·5g., $FeSO_4 \cdot 7H_2O$ : 0·01 g., Sucrose : 30·0 g., Agar : 15·0 g., Yeast extract : 1·0 g., Distilled water : 1000 ml.) agar enriched with 1% yeast extract. The pH of the medium was adjusted to 6·0 and temperature maintained at 25-27°C.

*Pathogenicity studies*:

Young spots of the strain TMV 2 were selected for collection of spores, since adequate amount of spores for inoculation purposes was not available from artificial cultures. The experimental plants were sprayed uniformly with the conidial suspension in water, incubated in moist-chambers for 48 hours and then kept in the green house where spots appeared within 15 days. In the field trials, the method of rating the severity of the disease was as follows: 100 leaflets with varying intensity of infection were col-

lected and the area of affected tissue expressed as percentage of the area of the entire leaflet, by actual measurement. Thus leaves with 2% to 6% infection were sorted out and maintained as standards for comparison with any sample collected from the field. (Plate II, Fig. 1).

*Varietal studies*:

All the different strains of *A. hypogaea* generally cultivated in South India TMV 2 (Tindivanum), TMV 3, TMV 4 and TMV 5 and a wild variety *A. rasteiro* (Husted, 1936) were tested for their susceptibility to the disease. *Sporulation* in the spots produced on the leaves was studied by first washing away the spores on a fresh young spot, drying the leaf on filter paper and then getting a new crop of spores under the moist chamber within 24 hours. The entire leaf (ves) were shaken up in 10 ml. sterile water and haemocytometer counts of the spores taken. The total infected area on the leaves was measured and then sporulation calculated as number of spores produced per sq. mm. of infected area. The different levels of humidity were maintained by using different concentrations of Analar $H_2SO_4$ and saturated salt solutions (Handbook of Chemistry and Physics 1953) and adjusted with the help of a humidometer.

Germination tests of spores were made on glass surface with spores suspended in a small drop of water. After 48 hours incubation period in a moist chamber the number of germinated spores were counted. The drop of water was allowed to dry under a fan before being kept in the moist chamber to form 'dry' spores, whereas the drop of water was retained to simulate dew or rainwater and forms the 'wet' spores.

*Survival of the fungus in soil*:

The following method was adopted to test the survival of the fungus on leaf material buried in soil. About 12—14 leaves of uniform size and infection (30%) were buried in small pots in 50g. of unsterilised garden soil. The moisture was adjusted to 20%, 40%, 60%, 80% and 100% and maintained throughout the course of the experiment. At the end of suitable incubation period a suspension of the entire lot of soil with the buried, decaying leaves were made in 200 ml. of sterile water. 30 ml. of the supernatant was then sprayed on five one-month old plants in one pot. Proper replicates were maintained wherever necessary. At the end of a fortnight, the number of spots per plant were counted and the average calculated.

*Symptomatology of the disease*:

The disease manifests itself in the field first as minute brown spots, with a yellow halo around, more prominent on the upper than on the lower surface. They are more or less round and are not delimited by the intervening veins. The petioles, stems and sometimes even the pegs are affected by long oval spots-girdling of the stem is seen in severe cases (Plate II, Fig. 2). Tufts of concentrically arranged conidiophores and conidia make their appearance on the undersurface presenting the characteristic brownish black colour. The diseased leaves curl themselves up and dry off prematurely (Plate II, Fig. 3).

## *Results*

*Morphology and growth in pure culture*:

Growth of the fungus is very slow taking about 6-7 days for a colony developing out of a single spore to be barely visible to the naked eye; and is peculiar with a raised centre and uneven folds, development being similar to that reported for *Cercoposora musae* (Calpouzos, 1955). The raised growth in the centre may be because of the slower rate of growth of the periphery of the colony when compared with that of its centre. The surface of these raised folds show different sectors with the aerial mycelia of different colours. The following types of growth in aerial mycelia are met with; (1) Brown, short, bristle like mass, which presents a deceptive appearance of dark brown conidia, the latter, however, being distinguished by their reddish colour in reflected light. (2) Patches of this brown sterile hyphae accumulate more of pigments as they grow older and become almost black. (3) Long, sparingly branched greyish white to white, thin, hyphae in direct contrast to the stout, short cells of the dark brown mycelia. Under all these types of aerial growth, the body of the colony is carbonaceous, constituted of dark brown cell compactly arranged and exuding a pinkish brown pigment into the surrounding medium. In about 5-6 weeks after inoculation, the colony reaches its maximum size of about 1 cm. in diameter. By selective sub-culturing it was possible to obtain colonies with only one type of aerial growth but under all conditions, the slow and isolated type of growth was evident. The fungus sporulates by two weeks after isolation, remains so for another fortnight, after which time sporulation is suppressed by the vegetative phase. It was observed that by selective subculturing of spores alone

at intervals of 2-3 weeks, the fungus could be maintained in culture in sporulating condition.

*Pathogenicity studies*:

Four of the 5 different strains of *Arachis hypogaea* generally cultivated in South India were tested for their susceptibility to infection under green house conditions. The results indicated that TMV 3 was significantly less susceptible than TMV 2, 4 and 5.

TABLE I

Incidence of disease in relation to leaf position using different varieties

| Varieties | Leaf position | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| TMV 2 .. | 36 | 32 | 47 | 41 | 62 | 23 | 6 |
| TMV 3 .. | 18 | 19 | 20 | 17 | 13 | 5 | 2 |
| TMV 4 .. | 38 | 40 | 46 | 56 | 59 | 49 | 16 |
| TMV 5 .. | 22 | 54 | 33 | 34 | 28 | 34 | 29 |

Analysis of data: (Table for variance)
Significance by 'F.' test Yes
Varieties: S.E.D.M. $\sqrt{387 \times 28 \times 2}$
At 5% level C.D. $147{\cdot}2 \times 2{\cdot}576 = 379$

| 1216 | 983 | 915 | 374 |
|---|---|---|---|
| TMV 4 | TMV 2 | TMV 5 | TMV 3 |

Leaf position: S.E.D.M. $\sqrt{387 \times 16 \times 2}$
at 5% level C.D. $113{\cdot}3 \times 2{\cdot}576 = 287$

| 643 | 593 | 881 | 580 | 454 | 440 | 197 |
|---|---|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 | 6 | 7 |

The data also reveals that the leaf that was oldest at the time of inoculation had the least number of spots indicating a physiological resistance, perhaps due to a depletion of starch. *Arachis rasteiro*, a wild species of spreading nature, showed high resistance when inoculated. The spots on this were characteristically minute, appearing as glossy, light brownish specks hardly producing any spore as against the bigger, black spots with the concen-

tric rings of condiophores and conidia found on the different strains of *A.hypogaea* (Table II).

TABLE II

Size of the spot produced on the different varieties and the sporulation of each

| Varieties used | Size of spot in sq.mm. (34 day old) Green house | No. of spores/sq.mm. (Green house sample) |
|---|---|---|
| TMV 2 .. | 3·5—6·7 × 3·2-6·1 | 38 |
| TMV 4 .. | 3·0—5·6 × 3·2-5·2 | 31 |
| TMV 3 .. | 3·0—5·1 × 3·5-4·9 | 10 |
| *A. rasteiro* .. | 3·0—4·3 × 2·5-4·0 | 5 |

*Environmental factors:*

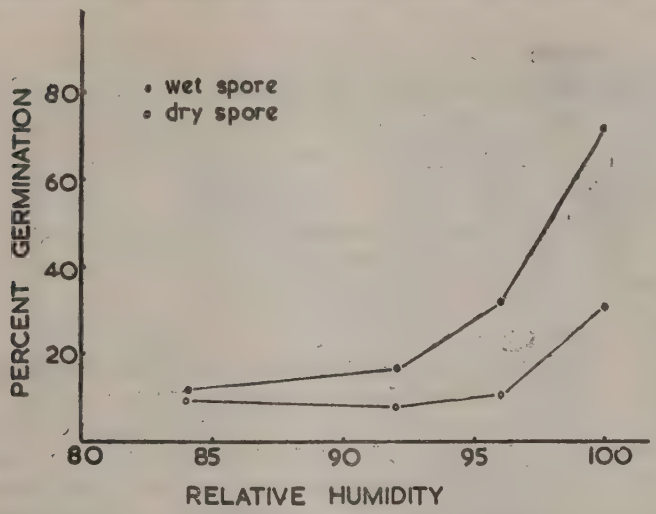
Influence of humidity on sporulation and spore germination: A relative humidity of 96-100 was necessary for good sporulation (Table III).

TABLE III

Effect of humidity on sporulation: No. of spores/sq.mm. (average temperature 77°F)

| % Humidity | Incubatoin period in hours | | | | % germination after 48 hours |
|---|---|---|---|---|---|
| | 24 | 48 | 72 | 96 | |
| 100 .. | 154 | 1507 | 827 | 723 | 89 |
| 96 .. | 91 | 718 | 227 | 198 | 12 |
| 93 .. | 83 | 75 | 60 | 47 | 3 |
| 88 .. | 42 | 40 | 17 | — | — |
| 80 .. | 15 | — | — | — | — |
| 73 .. | — | — | — | — | — |

Text-figure 1 shows that the germination percentage of 'wet' spores at 96% humidity was equal to that of 'dry' spores at 100




TEXT—FIG. 1. Effect of varying humidity on the germination of conidia of *C. personata*

indicating that conidia of *C.personata* germinate well at 96% humidity in the presence of dew or rain water.

*Maturity of crop*:

Field data showed that the short duration strain (TMV 2) was infected more than the long duration strains (TMV 3 & 4) of the same age (Table IV).

TABLE IV

Age of the crop and disease incidence as observed in the field (A.R.S., Tindivanam)

| | TMV 4 | TMV 2 | TMV 3 | *A. rasteiro* |
|---|---|---|---|---|
| Duration of crop .. | 135 | 105 | 135 | 135 |
| Age of the crop .. | 90–95 | 90–95 | 90–95 | 90–95 |
| % disease incidence .. | 10 | 30–45 | 10 | 1 |

Since the greenhouse trials (Table I) show that this cannot be a question of varietal susceptibility, the field data indicate that perhaps maturity of the crop may influence the development of

the disease. Further field observations tended to show that this may be more so if climatic conditions also happen to coincide with it.

*Age of leaves:*

The youngest and oldest leaves of the plant were found to be less susceptible (Table I) and this was correlated with the low carbohydrate content of these leaves (Shanta, 1957).

*Survival:*

The possible modes by which the pathogen can survive the severe summer are (a) to live in the soil or to be carried over by seeds and (b) to have an alternate host.

*Soil:*

It was observed that conidia of *C.personata,* being thin-walled, are unable to withstand long exposures to high temperatures. Since the perfect stage of the fungus was also not noticed on the infected leaves, it is assumed that the fungus survives as dormant mycelia on the infected debris of the previous crop. This assumption was strengthened by direct examination of soils collected from different localities with different climatic conditions and crop growth and also by their infectivity. The fungus survives in the soil on the fallen debris for as long as 22 weeks at 40% moisture level as was shown by a greenhouse experiment (Table V).

TABLE V

Survival of *C. personata* on infected leaves in the soil (Number of spots/plant, on spraying a 30 ml. of 1/4 soil suspension)

| Date of spraying | Incubation period in weeks | Moisture levels | | | | Remarks |
|---|---|---|---|---|---|---|
| | | 100% | 50% | 60% | 40% | |
| 13—7—55 | — | 9 | 9 | 9 | 9 | At 100 and 80% moisture levels some small dark brown spots appeared on the leaves probably caused by some soil bacteria. |
| 25—7—55 | 1 | 2 | 3 | 4 | 7 | |
| 1—8—55 | 2 | – | 2 | 2 | 5 | |
| 15—8—55 | 4 | – | – | 3 | 7 | |
| 29—8—55 | 6 | – | – | 2 | 5 | |
| 12—9—55 | 8 | – | – | – | 2 | |
| 26—9—55 | 10 | – | – | – | 2 | |
| 10—10—55 | 12 | – | – | – | 4 | |
| 24—10—55 | 14 | – | – | – | 3 | |
| 4—11—55 | 16 | – | 1 | – | 3 | |
| 21—11—55 | 18 | 1 | 6 | 6 | 4 | |
| 5—12—55 | 20 | 1 | 2 | 2 | 3 | |
| 19—12—55 | 22 | – | – | 1 | 1 | |

FIG. 1. Infected leaves collected from the field showing the standards for disease ratings. FIG. 2. Spots produced on leaves, petioles, pegs and the nut. FIG. 3. An infected leaf showing premature leafcurl and subsequent withering.

*Seed:*

It was noticed that only rarely does the pod get infected and therefore, the chance of infection of a crop through seed is negligible. Similarly, an all-out search for an alternate host gave negative results inasmuch as no weed common in the groundnut fields got infected by *C.perconata* and neither was groundnut attacked by other species of *Cercospora.*

## *Discussion*

*Cercospora personata,* causing the leaf spot disease of groundnut in South India, resembles many other members of this genus (Fernandez, 1953; Calpouzos, 1954; Nagel, 1934, Thomas, 1943) in its slow growth, shy sporulation and restricted host range.

All cultivated strains of groundnut (*A. hypogaea*) common in South India are susceptible. Duration of the crop seems to be an important factor in determining disease resistance or, rather, escapism. This is correlated with the age or maturity of a crop in the field. Thus the short duration crop TMV 2 matures in November-December, when the climatic factors are most favourable for the initiation and spread of the disease with optimum humidity and prolonged low temperature and dewy nights, and perhaps that is the reason why TMV 2 is the most susceptible in the field. Similar results were obained by Miller (1953) who correlated the maturation of the crop with disease susceptibility.

These results suggest that the short duration strains be planted as early in the season as possible so that the crop matures earlier and thereby escapes the climatic factors that is most conducive for the spread of the disease.

A relative humidity nearing 100 is necessary for good sporulation and for its successful establishment on the host. The spores germinate even at a R.H. 96% if a film of water is present. These factors show that when temperature and humidity conditions exist at any time to allow the formation of dew which, if retained sufficiently long for the spores to germinate and penetrate the host tissues, induces successful infection. That the climatic conditions of the winter months of South India are favourable for this is obvious and further detailed work on the formation of dew and its retention in the field in relation to other and similar factors of the micro-climate of the crop would give valuable results that would help in predicting an epidemic.

The fungus is not carried over on the seed, nor has it got any secondary host. This prompted Butler (1918) to suggest soil as a medium for survival. Since no perfect stage was observed in India, it is assumed that it is the dormant mycelia that persists in the leaves. That soil forms the source of inoculum for a fresh crop is evident from the fact that it is the older leaves that always get infected in the field, due, perhaps, to a proximity to the surface of the soil; although they are actually less susceptible to infection due to the depletion of starch.

*Acknowledgements*

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# Studies on 'Cercospora' Leaf Spot of Groundnuts (Arachis hypogaea)

## II. *Effect of N.P.K. fertilizers and certain fungicides on the incidence of the disease.**

BY

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(Received for publication on 30-8-1960)

ABSTRACT

Field trials on the effect of NPK fertilisers indicate that nitrogen and phosphorus significantly increase the disease incidence and potash dicreases it, though to a small extent.

Fungicidal treatments in the field reveal the efficiency of a mixture of red copper oxide and sulphur, Bordeaux mixture, sulphur and red copper oxide. Zinc-lime increases the disease which may, perhaps, be due to the effect of zinc on spore germination or due to its influence on the metabolism of the host.

Cercospora leaf spot of groundnuts is of common occurrence and is of considerable importance in the agricultural economy of this country. Very little work has been done here and elsewhere in spite of the importance of this disease. Much of the work pertains to the effect of several fungicides in controlling the disease and of the various sprays and dusts tested, the superiority of Bordeaux mixture was reported by McDougall (1941) and Van Hoof (1950). Most of the American Workers (Higgins, 1940; Woodroof, 1942; Miller, 1942; Simmonds, 1947) advocate the superiority of sulphur and copper-sulphur dusts over the others not only in controlling leaf spot but also in increasing the yield.

* Part of thesis approved for the Degree of Doctor of Philosophy of the University of Madras.

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## Material and Methods

*Fertiliser trial*:

*Layout of experiment*: The experiments were carried out in the field in properly replicated, randomised plots of 10′ × 8′ with a 2′ border row all round, the seeds (local spreading) being sown 9″ apart.

*Fertilisers*:

Ammonium sulphate, superphosphate and potassium sulphate were added at levels found to be best for yields of groundnut in Madras (John, *et al.* 1948) that is: (N) 20 lb. $(NH_4)_2SO_4$/acre = 20·6% N., (P) 40 lb. $CaH_4(PO_4)_2$/acre = 20% $P_2O_5$., (K) 60 lb. $K_2SO_4$/acre = 48% $K_2O$., (NP) 20·6% N + 20% $P_2O_5$., (NK) 20·6% N + 48% $K_2O$., (PK) 20% $P_2O_5$ + 48% $K_2O$. (NPK) 20·6% N, 20% $P_2O_5$ + 48% $K_2O$. The fertilisers were applied uniformly and incorporated into the soil one week before sowing the seeds.

*Fungicidal trials*:

*Layout of experiment*: The experiment was conducted in a randomised and replicated plot trial, with different treatments. The individual plot size was 9′ × 10′ with a border row of 1·5′ all round. The seeds (TMV 2) were sown 6″ apart.

*Fungicides*:

Red copper oxide, wettable sulphur, Bordeaux mixture, zinc sulphate + lime and copper oxide + sulphur were sprayed at concentrations (given as lb./gallon) of 1-30, 1-30, 5-5-50, 1-2-50 and 1-1-30. Three sprayings at 2-3 weeks intervals starting with six weeks after germination of seeds were given and observations on incidence of disèase were taken at the end of 2 and 3 months after sowing.

*Assessment of disease incidence*:

In the early stages of the disease, the actual number of spots per leaf per plant was noted. Counts on 20 plants taken at random per plot were made. In the advanced stage of the disease, the severity of infection was noted by comparison to the standards (Shanta, 1960).

*Laboratory trials*:

The effect of Zn $SO_4$ and CuO + S at different concentrations were tried in the laboratory on disease incidence and on spore

germination. The fungicides at the necessary concentrations were sprayed with an electric sprayer with the flow controlled at 100 ml/min. on test plants kept at 10 feet distance. The potted plants were uniformly rotated till a general run down of the solution was obtained. After 48 hr. incubation, the plants were inoculated and observed for incidence of disease. The method adopted to test the effect of the different fungicides on the germination of spors *in vivo* (leaf surface) was that of March (1936) with slight modifications.

## Results

*NPK trials:*

Phosphorus and nitrogen fertilisation significantly increased the disease incidence (Table I) and K-fertilisation, though decreased it, was not satisfactorily significant. Other treatments did not show any significant influence on disease incidence. These results agree in general with that found in wheat leaf rust (Chester, 1946) and other leaf diseases.

TABLE I

Effect of NPK fertilisers on disease incidence
(Graded in relation to leaf position)

| Leaf position | C | N | P | K | NP | NK | PK | NPK |
|---|---|---|---|---|---|---|---|---|
| 1 .. | 0 | — | — | — | — | — | — | — |
| 2 .. | 0·16 | 1·31 | 1·80 | 0·80 | 1·00 | 0·15 | 0·70 | 1·20 |
| 3 .. | 2·80 | 5·27 | 7·50 | 3·90 | 1·30 | 4·04 | 2·50 | 5·50 |
| 4 .. | 9·10 | 13·80 | 12·60 | 9·30 | 6·70 | 11·60 | 8·90 | 9·30 |
| 5 .. | 16·10 | 19·60 | 18·10 | 14·40 | 14·20 | 18·70 | 17·30 | 16·70 |
| 6 .. | 19·60 | 20·00 | 20·00 | 19·10 | 19·40 | 19·80 | 19·60 | 19·40 |

Analysis of variance:

Significance by 'F.' test = Yes

Treatments:

C. D. at 1% level $= \sqrt{\frac{2 \times 4{\cdot}88}{24}} \times 2{\cdot}604 = 1{\cdot}662$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10·0 | 9·99 | 9·05 | 8·66 | 8·16 | 7·97 | 7·93 | 6·96 |
| P | N | NK | NPK | PK | C | K | NP |

*Fungicidal trials:*

A mixture of red copper oxide and wettable sulphur (Table II) in the ratio 1:1:30 controlled the disease best while zinc—lime tended to increase it. All other fungicides decreased the disease incidence significantly the percentage of disease being 10·48, 9·33, 6·45 and 21·02 in sulphur, copper oxide and Bordeaux mixture sprayed and in control series respectively.

TABLE II

Incidence of disease as affected by fungicides
(Average of 4 plots — 20 plants in each plot)

| Fungicides used | First counts (2 months) | | Second counts (3 months) |
|---|---|---|---|
| | 24-9-'55 | | 27-10-'55 |
| | No. of spots per plant | % infection | % infection (field scale) |
| Red Cu0 .. | 19·3 | 73 | 9·33 |
| Wettable S .. | 14·4 | 54 | 10·48 |
| Bordeaux mixture .. | 15·3 | 58 | 6·45 |
| Zinc lime .. | 23·8 | 90 | 24·53 |
| CuO + S .. | 8·9 | 34 | 2·63 |
| Control .. | 26·5 | 100 | 21·08 |

Significance by F test = Yes (field scale)

Treatments:

C. D. at 5% level $\frac{288 \cdot 504}{25 \cdot 1} = 11 \cdot 49$

| | | | | | |
|---|---|---|---|---|---|
| 24·53 | 21·08 | 10·48 | 9·33 | 6·45 | 2·63 |
| Zn | Control | S | CuO | B. mix. | CuO + S |

When the effect of zinc sulphate and copper oxide with sulphur was tested separately on disease incidence and on spore germination, it was found (Table III) that zinc-lime at all concentrations tried (upto 750 p.p.m.) increased the disease incidence while CuO with sulphur in all the proportions tried completely controlled the disease. It was noticed that percentage germination of spores was highly increased in the presence of zinc while complete inhibition of same was noticed in the presence of the latter.

TABLE III

Effect of fungicides on disease incidence and on spore germination

| Treatments | Concentration | Comparative field concentration | A | B | | % germination |
|---|---|---|---|---|---|---|
| | | | No. of spots/ plant | Total No. of spores | No. germinated | |
| ZnSO4 | 100 p.p.m. | 0·22-50 | 44 | 92 | 63 | 68·5 |
| „ | 250 „ | 0·55-50 | 59 | 90 | 66 | 73·3 |
| „ | 450 „ | 1·00-50 | 59 | 98 | 69 | 70·4 |
| „ | 750 „ | 1·65-50 | 93 | 104 | 85 | 81·7 |
| „ | 1000 „ | 2·20-50 | 2 | 97 | *34 | 34·9 |
| CuO + S | 1:1 | 1:1:30 | — | 79 | 1 | 1·3 |
| „ | 2:1 | 2:1:30 | — | 90 | 2 | 2·2 |
| „ | 1:2 | 1:2:30 | — | 85 | 1 | 1·0 |
| Control | — | — | 56 | 88 | 21 | 23·9 |

*Many of the germinated spores had short germ tubes which failed to grow further.

## Discussion

The general effect of NPK fertilisation on disease incidence of 'tikka' disease agrees with many other leaf spot diseases that increase with nitrogen fertilisation. The importance of the role of nitrogen metabolism of the host in influencing the disease proneness of susceptible hosts is too well known to require any discussion and any such attempt with particular reference to this disease at this stage would be premature.

While Bordeaux mixture was undoubtedly good as a fungicide, the superiority of copper oxide + sulphur in the different proportions tried is self-evident. This effect of copper-sulphur sprays is in agreement with the many reports from America where it is widely used (Higgins, 1940, Miller, 1942) not only because of its fungicidal properties but also because of its nutritional value.

Although zinc-lime is reported to be a good fungicide (Roberts and Pierce, 1932) it was found to increase the disease incidence in the case of 'tikka'. Perhaps this may be due to an increased percentage germination of spores in the presence of zinc as shown here and elsewhere (Shanta, 1957). Evidence for this stimulatory effect of zinc on spore germination is not lacking, for, it is known that zinc along with an unknown 'gelatin factor' has favourable effect on the lyophilisation and germ tube formation of the uredospores of Puccinia (Sharp, 1954). Viewed in this light, studies on the interaction between some of these heavy metals and 'growth factors' are bound to yield interesting results both in the fundamental and applied fields.

## Acknowledgement

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# On the Isomorphism between rotations in Three-space and a Complex Two-space

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ABSTRACT

It is shown that rotations in three space induce corresponding transformations in a complex two-space. Corresponding to every direction in 3-space, there is a vector and a matrix in 2-space. The relationship of this to the representation of the polarisation states of light beams is discussed.

## 1. *Introduction*:

In this paper we shall deal with methods of representing rotations in ordinary three-space by suitable transformations in a two-space, the complex plane. Any rotation in three-space requires three parameters for describing it, two to denote the direction of the axis of rotation and one to give the angle of rotation. However, these three parameters may all be angles as in the well-known Eulerian angles, which are the magnitudes of rotations about specified directions. And similarly, the direction of the axis of rotation may be denoted by either the direction cosines $(l, m, n)$ of which two are independent or the spherical polar angles $(\theta, \varphi)$.

It is known (Whittaker, 1952) that any rotation can be specified by four parameters, known as the Cayley-Klein parameters $\alpha, \beta, \gamma, \delta$ which, however, are related by the equation $(\alpha\delta - \beta\gamma) = 1$. Further, these parameters may be arranged in the form of $(2\times2)$ matrix $\mathbf{Q} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and then, there is a one-to-one correspondence between the matrix $\mathbf{Q}$ and the matrix of rotation $\mathbf{A}$ in three-space. In three-space, the effect of rotation is expressed by the equation $X' = \mathbf{A}X$ where $X$ stands for the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. In two space, the corresponding equation in terms of $\mathbf{Q}$ can be written in two differ-

ent ways as will be shown in this paper. We can either represent a direction $X$ in three-space by a matrix $\mathbf{P}$ in two-space in which case the rotation is represented by $\mathbf{P}' = \mathbf{Q}\mathbf{P}\mathbf{Q}^{-1}$ or we may represense $X$ by a vector $U = \begin{pmatrix} u \\ v \end{pmatrix}$ in two-space when the rotation is represented by $U' = \mathbf{Q}U$. In either case, it is interesting to note that the matrix $\mathbf{Q}$ is the same. The matrix $\mathbf{P}$ has a close relationship to the Pauli matrices and may in fact be written as $\mathbf{P} = X \cdot \sigma$ where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. So also, the relationship between the vector $U$ in two-space and the vector $X$ in three-space may be written in the form $X = U^{+}\sigma U$. Both these relations have a physical analogy in the relation between the resolved components of an elliptically polarised beam of light along two orthogonal directions of its wave-front and its representative point on the Poincaré sphere. This paper deals with the inter-relations between these interesting aspects of representing the operation of rotation by matrices. The attempt has been to make the discussion as simple as possible and in view of this, the group-theoretical aspects of the problem are not touched upon. An account of this aspect will be found in the article by Eckart (1930). An account of the Cayley-Klein parameters is also found in Whittaker (1952) and Goldstein (1953).

## 2. *Repesentations of Rotation in Three-space:*

We shall denote a direction in three-space by the three-components $(x, y, z)$ of a unit vector parallel to that direction such that $x^2+y^2+z^2=1$. The vector is denoted by the symbol $X$. Let us rotate the coordinate axis about an axis with direction cosines $(l, m, n)$, through an angle $\omega$. The new direction $X'$ is given by

$$X' = \mathbf{A}X \qquad .. \quad (1)$$

where $\mathbf{A}$ is a $(3 \times 3)$ matrix whose elements are functions of $l$, $m$, $n$ and $\omega$. Of the different representations fo the matrix $\mathbf{A}$, we shall adopt only two.

Define four complex parameters $\alpha$, $\beta$, $\gamma$, $\delta$ by

$$\left.\begin{aligned} \alpha &= \cos\frac{\omega}{2} + i n \sin\frac{\omega}{2} = \delta^* \\ \beta &= i(l - i m) \sin\frac{\omega}{2} = -\gamma^* \end{aligned}\right\} \qquad .. \quad (2a)$$

We note that $\alpha\delta - \beta\gamma = 1$ .. (2b)

In terms of (1) we can write **A** as

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}(\alpha^2-\beta^2-\gamma^2+\delta^2) & \frac{i}{2}(-\alpha^2-\beta^2+\gamma^2+\delta^2) & -(\alpha\beta-\gamma\delta) \\ \frac{i}{2}(\alpha^2-\beta^2+\gamma^2-\delta^2) & \frac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) & -i(\alpha\beta+\gamma\delta) \\ -(\alpha\gamma-\beta\delta) & i(\alpha\gamma+\beta\delta) & \alpha\delta+\beta\gamma \end{pmatrix} \quad .. \quad (3)$$

These four parameters $\alpha$, $\beta$, $\gamma$, $\delta$ are known as the Cayley-Klein parameters.*

However, the Cayley-Klein parameters may be expressed in an elegant form in terms of the Eulerian angles, $\psi$, $\chi$, $\Phi$. In terms of these the matrix **A** takes the form

$$\mathbf{A} = \begin{pmatrix} \cos\psi\cos\Phi - \cos\chi\sin\Phi\sin\varphi & -\sin\varphi\cos\Phi - \cos\chi\sin\Phi\cos\varphi & \sin\chi\sin\Phi \\ \cos\psi\sin\Phi + \cos\chi\cos\Phi\sin\varphi & -\sin\varphi\sin\Phi - \cos\chi\cos\Phi\cos\varphi & -\sin\chi\cos\Phi \\ \sin\chi\sin\psi & \sin\chi\cos\psi & \cos\chi \end{pmatrix} \quad .. \quad (4)$$

The relation between $\alpha$, $\beta$, $\gamma$, $\delta$ and $\psi$, $\chi$, $\Phi$ can be obtained by equating the terms in (3) and (4) and they are given by

$$\left.\begin{aligned} \alpha &= \cos\tfrac{1}{2}\chi\,\exp\tfrac{1}{2}i\,(\psi+\Phi) = \delta^* \\ \beta &= i\sin\tfrac{1}{2}\chi\,\exp\tfrac{1}{2}i\,(\Phi-\psi) = -\gamma^* \end{aligned}\right\} \quad .. \quad (5)$$

It may be noticed that while the actual rotations in three-space are through angles, $\psi$, $\chi$, $\Phi$, these parameters contain only half angles.

* The definition adopted here for the Cayley-Klein parameters as well as the Eulerian angles agree with that of Goldstein but differs slightly from that of Whittaker. As will be seen below, this definition leads to elegant relations with other well-established conventions (e.g.) the definition of the three-Pauli matrices.

3. *Representation of a Direction by a Complex Variable and a Two-vector U*:

(i) Let a point P $(x, y, z)$, be stereographically projected on the $x$-$y$ plane. Let this point be Q. The coordinates of Q in terms of the spherical polar angles $(\theta, \varphi)$ of P (fig. 1) are

$$x_Q = \tan \frac{\theta}{2} \cos \varphi; \; y_Q = \tan \frac{\theta}{2} \sin \varphi \qquad \ldots \quad (6)$$

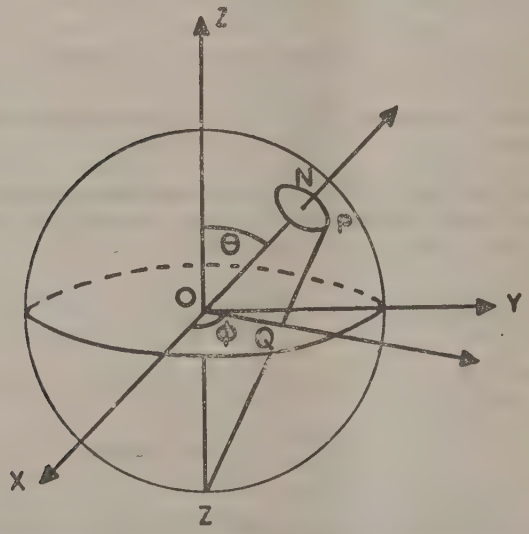


Fig. 1. Stereographic projection of a point: The stereographic projection of P on the sphere is Q on the x-y plane

We can define a complex variable Z as

$$Z = x_Q + iy_Q = \tan \frac{\theta}{2} \exp (i\varphi) \qquad \ldots \quad (7)$$

Since we have a unique point Q on the $x$-$y$ plane corresponding to every direction, it is possible to associate a complex variable Z with every direction.

Under rotation of axis, the transformation of Z is homogaphic (Whittaker, 1952) and can be put as

$$Z' = \frac{pZ+q}{rZ+s} \qquad \ldots \quad (8)$$

The law of combination of these parameters $p$, $q$, $r$ and $s$ can be found using the properly that the transformation Z - - - Z″ is equivalent to the two successive transformation Z - - - Z′ and Z′ - - - Z″. Defining a matrix **R** by

$$\mathbf{R} = \begin{pmatrix} s & r \\ p & p \end{pmatrix} \qquad \ldots \quad (9)$$

We can write the combination law as

$$\mathbf{R} = \mathbf{R}_2 \mathbf{R}_1 \qquad \ldots \quad (10)$$

In term of the matrix A, this transformation corresponds to

$$\mathbf{A} = \mathbf{A}_2 \mathbf{A}_1$$

It is possible to show that $p$, $q$, $r$, $s$ are identical with $\delta$, $\gamma$, $\beta$, $\alpha$ and obey the same laws of transformation as these (Whittaker 1952).

In the pevious sub-section, we saw how we can represent a direction by using a single complex variable Z. It is possible to do the same, using a pair of complex variables $v$ and $u$ (Whittaker, 1952, Walker, 1956, Jeffreys and Jeffreys, 1950). However, it must be noted that in the former case Z has just two independent components, while a direction in three-space also requires two parameters for its specification. When two complex variables $u$ and $v$ are used, there are four parameters involved and consequently to get uniqueness there must be two relations between them. This is best done by considering $u$ and $v$ to be normalised by the equation

$$|u|^2 + |v|^2 = 1 \qquad \ldots \quad (11)$$

and considering that only the relative phase difference between $v$ and $u$ is significant. This would mean that the phase of $u$ can be taken to be zero, or any arbitrary fixed value, without affecting the discussion below (see also Section 6).

Let us define the two complex variables $u$, $v$ by

$$v/u = \mathrm{Z} \qquad \ldots \quad (12)$$

using (7) and (12) we can show that

$$\left.\begin{aligned} x &= uv^* + u^*v \\ y &= i(uv^* - u^*v) \\ y &= uu^* - vv^* \end{aligned}\right\} \qquad \ldots \quad (13)$$

This can be put in a very compact form if we use the Pauli spin matrices defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (14)$$

Using this representation, we find that

$$X = U^{+}\sigma U \qquad .. \quad (15)$$

Where $U$ represents the column vector

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \text{ with } U^{+}U = 1 \qquad .. \quad (16)$$

Thus the 2-vector $U$ may be taken to represent a direction $X$ in 3-space. $u$, $v$ are the two components of a spinor (Brinkman, 1956). The latter is then explicitly given in terms of $U$ through the Pauli matrices. The components of $X$ in any coordinate system are given by (13).

(b) We shall now examine the transformation properties of $U$. To do this, let us define a Q-matrix

$$\mathbf{Q} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \text{ det } \mathbf{Q} = +1 \qquad .. \quad (17)$$

Where, for the present, we do not assume that $\alpha$, $\beta$, $\gamma$, $\delta$ are the same as the Cayley-Klein parameters.

If $U'$ represents the new (transformed) direction then,

$$U' = \mathbf{Q}U \qquad .. \quad (18)$$

(i.e.) the x', $y'$, $z'$ obtained from $u'$, $v'$ are just the same as would be obtained from (3).

Therefore using (13)

$$\begin{aligned} x' &= \tfrac{1}{2}(\alpha^2-\beta^2-\gamma^2+\delta^2)x + \tfrac{1}{2}i(-\alpha^2-\beta^2+\gamma^2+\delta^2)y + (\gamma\delta-\alpha\beta)z \\ y'' &= \tfrac{1}{2}i(\alpha^2-\beta^2+\gamma^2-\delta^2)x + \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2)y + (-\alpha\beta-\gamma\delta)z \\ z'' &= (\beta\delta-\alpha\gamma)x + i(\alpha\gamma+\beta\delta)y + (+\alpha\delta+\beta\gamma)z \qquad .. \quad (19) \end{aligned}$$

Comparing with (3) we see that the elements of the matrix $\mathbf{Q}$ are in fact the Cayley-Klein parameters.

A further confirmation of this is obtained by considering the relation (7) between the components $u$, $v$ of $U$ and the parameter Z. Substituting in (8) we have,

$$\frac{v'}{u'} = \frac{pv + qu}{rv + su} \qquad .. \quad (20)$$

from (18) we have

$$\frac{v'}{u'} = \frac{\delta v + \gamma u}{\beta v + \alpha u} \qquad .. \quad (21)$$

It is readily verified that (20) and (21) are consistent with the relations mentioned at the end of section 3.

4. *Representation of a Direction by a Complex* (2×2) *Matrix* **P**:

It is also possible to associate a complex (2×2) matrix **P** with a direction in three-space. We define the matrix **P** as

$$\mathbf{P} = X \cdot \sigma = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}; \text{ det } \mathbf{P} = -1 \qquad .. \quad (22)$$

If $\mathbf{P}'$ represents the new (transformed) matrix after the rotation (i.e.)

$$\mathbf{P}' = X' \cdot \sigma \qquad .. \quad (23)$$

then we can show that

$$\mathbf{P}' = \mathbf{QPR}^{-1} \qquad .. \quad (24)$$

Where **Q** is the same matrix as in the previous section, which corresponds to rotation in three-space.

To show this generally, we notice that

$$\mathbf{Q}^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

So, we have

$$\mathbf{QPQ}^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \mathbf{R}, \text{ say} \qquad .. \quad (25)$$

Explicitly written out

$$\left.\begin{aligned} R_{11} &= (-\alpha\gamma+\beta\delta)\,x+i\,(\alpha\gamma+\beta\delta)\,y+(\alpha\delta+\beta\gamma)\,z \\ Rs_{12} &= (\alpha^2-\beta^2)\,x-i\,(\alpha^2+\beta^2)\,y-2\alpha\beta z \\ R_{21} &= (-\gamma^2+\delta^2)\,x+i\,(\gamma^2+\delta^2)\,y+2\gamma\delta z \\ R_{22} &= (\alpha\gamma-\beta\delta)\,x-i\,(\alpha\gamma+\beta\delta)\,y-(\alpha\delta+\beta\gamma)\,z \end{aligned}\right\} \quad .. \quad (26)$$

Using equation (3) we immediately obtain

$$\begin{aligned} R_{11} &= z', & R_{12} &= x'-iy'; \\ R_{21} &= x'+iy'; & R_{22} &= -z'. \end{aligned} \quad .. \quad (27)$$

In other words, the R-matrix is just the matrix $\mathbf{P}'$. Thus, we have established (24) in the general case.

5. *Discussion of the various Isomorphism:*

We have established above, a number of correspondences between rotations in three-space and transformations in a two-space, which may be put in the form of a table.

Also for ready reference, we give below the three Q-matrices explicitly

$$\mathbf{Q}_x = \begin{pmatrix} c & is \\ is & c \end{pmatrix}; \quad \mathbf{Q}_y = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}; \quad \mathbf{Q}_z = \begin{pmatrix} c+is & o \\ o & c-is \end{pmatrix} \quad .. \quad (28)$$

These are also related to the Q-matrices for the Eulerian rotations as follows

$$\mathbf{Q}_\varphi = \mathbf{Q}_\psi = \mathbf{Q}_z; \qquad \mathbf{Q}_\theta = \mathbf{Q}_x \quad .. \quad (29)$$

It will be seen that in all the three representations the effect of a rotation in three-space is expressed in the two space *via.* the Cayley-Klein parameters, $\alpha$, $\beta$, $\gamma$, $\delta$. Taking the cases (b) and (c), in the fomer, a vector in two-space represents a direction in three-space, while in the latter, a matrix $\mathbf{P}$ in two space represents the same. In either representation, the effect of a rotation in three-space corresponds to the operation of a $(2 \times 2)$ matrix $\mathbf{Q}$, (ie.) $\mathbf{Q}$ operates either on the vector $U$ to give $U' = \mathbf{Q}U$ or on the matrix $\mathbf{P}$ to give $\mathbf{P}' = \mathbf{QPQ}^{-1}$. This relationship between the matrix and vector representation is by no means obvious and it was not at all clear at the beginning of this investigation that the rotation operator $\mathbf{Q}$ will be te same for both. The possibility of representing the orthogonal matrices in three-space by $(2 \times 2)$

TABLE I

*Summary of Main Results*

| Representation of a Direction $X$. | Representation of a Rotation $(A)$ | Relation between $X$ & its representation | Relation between Cayley-Klein Parameters and the representation of Rotation. |
|---|---|---|---|
| (a) A complex variable Z | $Z' = \frac{pZ + q}{rZ + s}$ | $Z = x_Q + iy_Q$ $= \tan \frac{1}{2}\theta \exp(i\varphi)$ where $(x_Q \; y_Q)$ is the stereographic projection of $X$. | $p = \delta, \; q = \gamma$ $r = \beta, \; s = \alpha$ |
| (b) A two vector $U$ $U = \binom{u}{v}$; $\lvert u\rvert^2 + \lvert v\rvert^2 = 1$ | $U' = \mathbf{Q}U$ $\det \mathbf{Q} = +1$ | $\frac{v}{u} = \tan \frac{1}{2}\theta \exp(i\varphi)$ $X = U^+ \sigma U$ | $\mathbf{Q} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ |
| (c) A $2 \times 2$ matrix $\mathbf{P}$ $\det \mathbf{P} = -1$ | $\mathbf{P}' = \mathbf{QPQ}^{-1}$ | $\mathbf{P} = X \cdot \sigma$ | $\mathbf{Q}$ same as in (b) |

unitary matrices is well-known (Jeffreys, 1950; Weyl, 1930). However, a $(2 \times 2)$ matrix has four independent parameters. As will be shown below, one of these paramters appears as merely a phase factor exp $i\varepsilon$, where $\varepsilon$ could by definition be made zero and then the correspondence becomes one-to-one.

Thus the question naturally arises as to whether corresponding to a given **A** we have a unique **Q**. To answer this, let us consider the configuration in three space after the Eulerian angles are continuously changed by $2\pi$. This gives the same state in real space. However, we find that the Cayley-Klein parameters, $\alpha$, $\beta$, $\gamma$, $\delta$ all change sign since they are functions of half-angles. Thus, the same relation **A**, may be represented by two matrices, viz., **Q** and also by **Q'**.

$$\mathbf{Q}' = -\mathbf{Q}; \text{ det } \mathbf{Q} = +1 \quad \ldots \quad (30)$$

This double isomorphism arises essentially because the parameters involve only half-angles. This should however, lead to no difficulty in practice. The two representations **Q**, $-\mathbf{Q}$ are similar to the two branches of the square-root of a complex variable.

As mentioned above, in both the transformations defined in equations (18) and (24), the Q-matrix is arbitrary to the extent of a factor exp $i\varepsilon$ where $\varepsilon$ is real, although, the arbitariness has been avoided in those equations by the condition that det $\mathbf{Q} = +1$. Thus, we could use a continuous infinity of Q-matrices obtained from that defined by (18) by multiplying it with exp $i\varepsilon$. Thus the actual correspondence **A** and **Q** using this idea alone is one-continuous infinity. However, if a particular value of $\varepsilon$ is chosen, then it would obviously be taken to be the same for all rotations, i.e., all $\mathbf{A} \longrightarrow \mathbf{Q}$ exp $i\varepsilon$. It is obvious that the isomorphism $\mathbf{A} \longrightarrow -\mathbf{Q}$ is only a special case of this when $\varepsilon = \pi$.*

We can further show that the only matrices that are equivalent to **A** are **Q** exp $i\varepsilon$ and no others. To show this, let us suppose that **S** is another matrix such that

$$\mathbf{P}' = \mathbf{SPS}^{-1} \quad \ldots \quad (31)$$

* The different values, of $\varepsilon$ correspond to changes in the absolute phase of the light wave in the optical applications discussed below. They do not affect the relative phase retardation, or the state of elliptically polarised wave.

for all **P**. Since **P** is Hermitian, both **S** and **Q** are unitary and we can show that $\mathbf{S} = k\mathbf{Q}$ .. (32)

Now, $\det \mathbf{S} = \det k\mathbf{Q} = k^2 \det \mathbf{Q} = k^2$ using (17).

$$\therefore k = \exp i\varepsilon \text{ or } \mathbf{S} = \mathbf{Q} \exp i\varepsilon \quad .. \quad (33)$$

We note however that, we may relax the condition that $\det \mathbf{Q} = +1$ and only require that $|\det \mathbf{S}| = 1$.

The representation (a) of Table I, which is the analytic representation of the streographic projection, was made use of by Poincaré to represent the States of polarisation of a light wave. He employed the inverse of our correspondence. The state of polarisation may be represented by a complex number Z, which could then be related to the direction in three-space or to a point on a unit sphere — the so-called Poincaré sphere. We shall consider this briefly in the next section, where it will also be shown that the relation $X = U^{\dagger} \sigma U$ is equivalent to the relation between the state of polarisation of a light wave and its Stokes parameters. A detailed account of Poincaré sphere and of Stokes parameters may be obtained from the article of Ramachandran and Ramaseshan (in press). See also Ramachandran (1952) and Walker (1954).

6. *Representation of the state of a Polarised beam of Radiation on the Poincaré Sphere*:

The state of polarisation of a beam of light can be specified by the azimuth $\lambda$ and the ellipticity $\varepsilon$ of the elliptic vibration (fig. 2), where we define $\tan \varepsilon = b/a$. $\varepsilon$ being + ve for left rotation ellipses, and — ve for right rotating ellipse. The range of variation of the parameters $\lambda$, $\varepsilon$ are

$$0 \leqslant \lambda \leqslant \pi; \quad -\tfrac{1}{4}\pi \leqslant \varepsilon \leqslant +\tfrac{1}{4}\pi \quad .. \quad (34)$$

Consider a simple harmonic vibration to be represented by $\exp +i(\omega t + \alpha)$ where the positive sign is chosen for the exponential* and $\alpha$ is the phase, then the resolved component of an elliptic

* There seems to be no uniformity at all among the authors regarding the sign convention for representing vibrations by exponentials. The definition adopted here seems to be most suitable and it agrees with that adopted by most writers on Optics although it is contrary to that employed in Quantum Mechanical Literature, where there is however fortunately a uniformity.

vibration of unit intensity along OA and OB will be cos ε and $-i \sin \xi$.

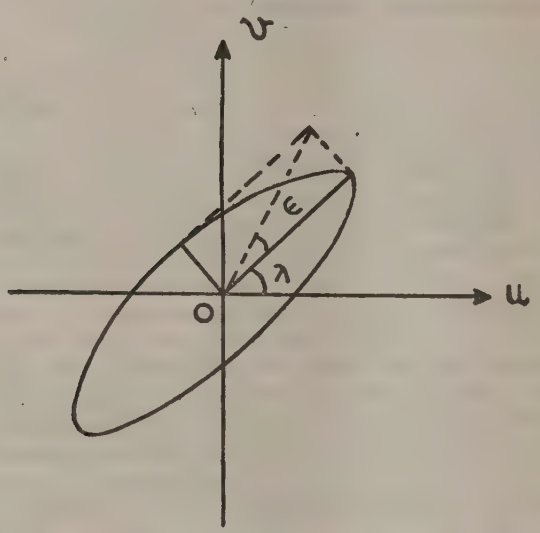


FIG. 2. Representation of the states of polarisation of a light beam

Then it follows (Ramachandran and Ramaseshan, in press) that the components of the elliptic vibration along OX and OY are

$$\left.\begin{aligned} u &= \cos\ \varepsilon\ \cos\ \lambda + i\ \sin\ \varepsilon\ \sin\ \lambda \\ v &= \cos\ \varepsilon\ \sin\ \lambda - i\ \sin\ \varepsilon\ \cos\ \lambda \end{aligned}\right\} \quad .. \quad (35)$$

From these, it is readily shown that the phase difference δ between $u$ and $v$ and the ratio of their meanings $|v|/|u|$ are given by

$$\left.\begin{aligned} \tan\ \delta &= \tan\ 2\varepsilon/\sin\ 2\lambda\ (=S/C) \\ \cos\ 2\alpha &= \cos\ 2\varepsilon\ \cos\ 2\lambda\ (=M) \end{aligned}\right\} \quad .. \quad (36)$$

where

$$|v|/|u| = \tan\ \alpha \quad .. \quad (37)$$

and obviously $|u|^2 + |v|^2 = 1$.

So also, it can be shown from (35) that

$$\left.\begin{aligned} uu^*-vv^* &= \cos 2\varepsilon \cos 2\lambda = M \\ uv^*+u^*v &= \cos 2\varepsilon \sin 2\lambda = C \\ i(uv^*-u^*v) &= \sin 2\varepsilon \quad\; = S \end{aligned}\right\} \qquad .. \quad (38)$$

Here, the quantities $\cos 2\varepsilon \cos 2\lambda$, $\cos 2\varepsilon \sin 2\lambda$ and $\sin 2\varepsilon$ are the Stokes parameters M, C, S respectively of a completely polarised beam. It will be noted that these are closely related to $\alpha$ and $\delta$, as in equation (36).

Thus we have the following relations:

The state of a polarised beam may be specified by

(1) its azimuth $\lambda$ and ellipticity $\varepsilon$,

(2) its components $u$, $v$, along the axes OX and OY lying on the wave front, and hence by a complex 2-vector $U=\begin{pmatrix} u \\ v \end{pmatrix}$ where by definition we may take the phase of $u$ to be zero;

(3) its Stokes parameters M, C, S, given by equation (38).

It is readily seen that M, C, S, are the components of a unit-vector in hypothetical 3-space (the Poincaré Space) along the

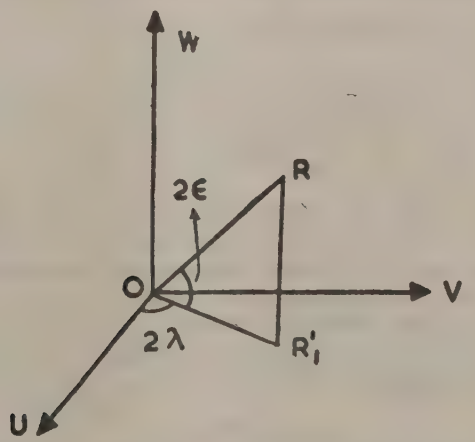


Fig. 3. Poincare space and the coordinate axes U. V. W

three orthogonal axes, OU, OV, OW where $(2\lambda, 2\varepsilon)$ are the longitude and the latitude of the direction corresponding to the vector with respect to these axes, (fig. 3).

Thus all states of polarisation may be represented by points on a unit-sphere in the Poincaré space (the so-called Poincaré sphere); since $2\lambda$, $2\varepsilon$ will now cover the range

$$O \leqslant 2\lambda \leqslant 2\pi; \quad -\tfrac{1}{2}\pi \leqslant 2\varepsilon \leqslant +(\pi/2) \qquad .. \quad (39)$$

At the same time, it follows that the orthogonal components of the Poincaré vector are the Stokes parameters.

This method of introducing the Poincaré sphere seems to be superior to that of Ramachandran and Ramaseshan (loc. cit.), where the relationship between the latitude $2\varepsilon$ and the longitude $2\lambda$ of a point on the Poincaré sphere and the azimuth ($\lambda$) and the ellipticity ($\varepsilon$) of the elliptic vibration is taken to be given by a definition. Pockels (1906) following Poincaré, has shown the relationship through the stereographic projection, but in this the relation to the Stokes parameters is not at all clear.

It may be mentioned that the Stokes parameters corresponding to the state represented by $U$ has a close relationship to the 'density-matrix' (Falkoff, 1951) defined by

$$\mathbf{P} = (P_{ij}) = UU^{+} \qquad .. \quad (40)$$

It is obvious that if we take that

$$uu^* + vv^* = 1$$

then

$$\left.\begin{aligned} M = P_{11} - P_{22} &= uu^* - vv^* \\ C = P_{12} + P_{21} &= uv^* + u^*v \\ S^* = -S = i(P_{21} - P_{12}) &= i(u^*v - uv^*) \end{aligned}\right\} \qquad .. \quad (41)$$

are exactly of the same form as the relationship between the vector $U$ in the $(u, v)$ plane and the components $x, y, z$, in the 3-space in equation (13). Note, however, that they are not exactly equal to the Stokes parameters, but that the third quantity is equal to $-S$. This change of sign occurs because of the sign convention of the exponential form $\exp i\omega t$. Equation (13) as already shown is equivalent to

$$X = U^{+}\sigma U$$

and the matrices have been defined so as to agree with the commutation relations of the angular momentum operator in Quantum Mechanics. The change is that in the latter the exponential form is $\exp -i\omega t$, which is the standard convention in Quantum

Mechanics. The change of this to exp $+i\omega t$ would alter only the last equation in (41) which alone contains the factor $i$.

7. *Relation between the Q-Matrix and Jones' Matrices for Polarisation states:*

It can be shown (Ramachandran and Ramaseshan, in Press) that the effect of the passage of polarised light through a transparent birefringent medium of the most general type is equivalent to a rotation in Poincaré space. It then follows from the correspondence (b) of Table I that this must be equivalent to the operation of a $(2 \times 2)$ unitary matrix on the state vector $U$.

This can be shown in a straightforward manner as follows (Jones, 1941). Let $\varphi_1$, $\varphi_2$ be the phase retardations introduced in the two components $u$, $v$ parallel to the principal axes of the crystal normal to the direction of propagation. Then

$$u' = u \exp(-i\varphi_1); \quad v' = v \exp(-i\varphi_2) \qquad .. \quad (42)$$

and

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \exp(-i\varphi) \begin{pmatrix} \exp(-i\gamma) & 0 \\ 0 & \exp(i\gamma) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad .. \quad (43)$$

where $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2); \quad \gamma = \frac{1}{2}(\varphi_1 - \varphi_2)$ .. (44)

Since exp $(i\varphi)$ alters only the absolute phase, and does not introduce any relative phase difference, we may write (43) in the following form, setting $\gamma = \frac{1}{2}\triangle$ where $\triangle$ is the relative phase retardation:

$$U' = \mathbf{G}U, \text{ where } \mathbf{G} = \begin{pmatrix} \exp(-\frac{i}{2}\triangle) & 0 \\ 0 & \exp(\frac{i}{2}\triangle) \end{pmatrix} \qquad .. \quad (45)$$

It is remarkable that this matrix is identical with Z, but for the change in sign of the angle $\triangle$. In a similar manner, the Jones' matrices corresponding to rotations about the other two axes of the Poincaré space are

$$\mathbf{G}' = \begin{pmatrix} \cos\frac{\triangle}{2} & -i\sin\frac{\triangle}{2} \\ -i\sin\frac{\triangle}{2} & \cos\frac{\triangle}{2} \end{pmatrix}; \quad \mathbf{S} = \begin{pmatrix} \cos\frac{1}{2}\triangle & -\sin\frac{1}{2}\triangle \\ \sin\frac{1}{2}\triangle & \frac{1}{2}\cos\triangle \end{pmatrix} \qquad .. \quad (46)$$

Here again, they are seen to be the same as $Q_x$ and $Q_y$, but for the fact that the argument for the angle is negative. This negative sign, as already explained, arises because in optical studies, the exponential time factor is taken in the form $\exp + i\omega t$ and the phase is of the form $\exp(-i\varphi)$.

This equivalence of the Jones' matrices with the Q-matrices is consistent with the Poincaré representation. The interesting fact is that the $2 \times 2$ matrices here represent operations in real space (namely on the vibration vectors in the plane at right angles to the direction of propogation), while the corresponding 3-space is the hypothetical Poincaré space. In this 3-space, all changes in the polarisation states of a light beam can be represented by real rotations, i.e., by $3 \times 3$ matrices with real components. This is in essence the reason why the Poincaré representation is found to be convenient for dealing with polarised light.

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